Quiz 5

1. Let S be a nonempty subset of $\mathbb{R}$ which is bounded above and below, and let $m = \inf S$. Prove that $m \in S$ if and only if $m = \min S$.

   Solution. This is an if-and-only-if statement. Thus, in order to prove it, we need to show two implications (going in the opposite directions).

   First, assume that $m = \min S$. Then, by definition of a minimum[1], m must be an element of S.

   Conversely, suppose that m is an element of S. Since m is the infimum of S, then it is also a lower bound[2] of this set. Then, by definition of a minimum, $m = \min S$.

2. Prove: if $x < y$ are real numbers with $x < y$, then there are infinitely many rational numbers in the interval $[x, y]$.

   Solution. Clearly, it suffices to show that the open interval (x, y) contains infinitely many rationals. We will prove this statement by contradiction. Suppose that the set S of all rational numbers contained in the interval (x, y) is finite. Theorem 12.12 guarantees that this set is not empty. Thus, we can pick the smallest element of S. Let us denote it by r. Now, by theorem 12.12, there exists a rational number r' such that $x < r' < r$. This contradicts the choice of r as the *smallest* rational number contained in the interval (x, y). Q.E.D.

[1] See definition 12.2.
[2] See definition 12.5.