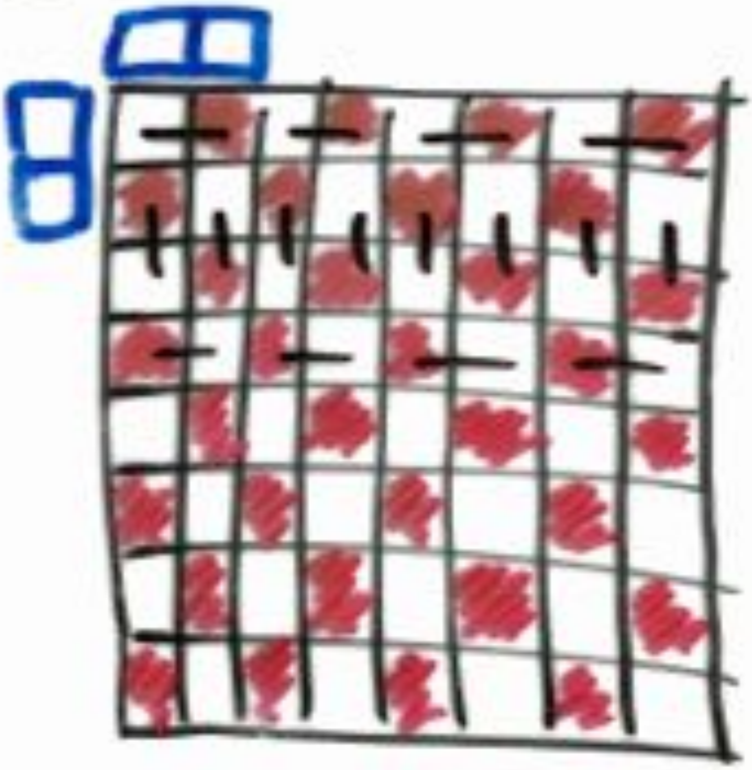
§§3-4 Techniques of Proof

"Proof" and "Theory" (or "Theorem") have very different meanings in mathematics, compared to other fields.

Ex Chessboard Tiling.

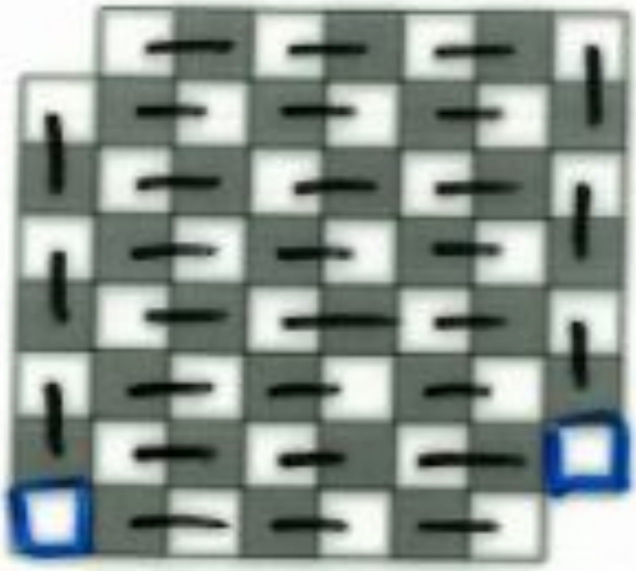


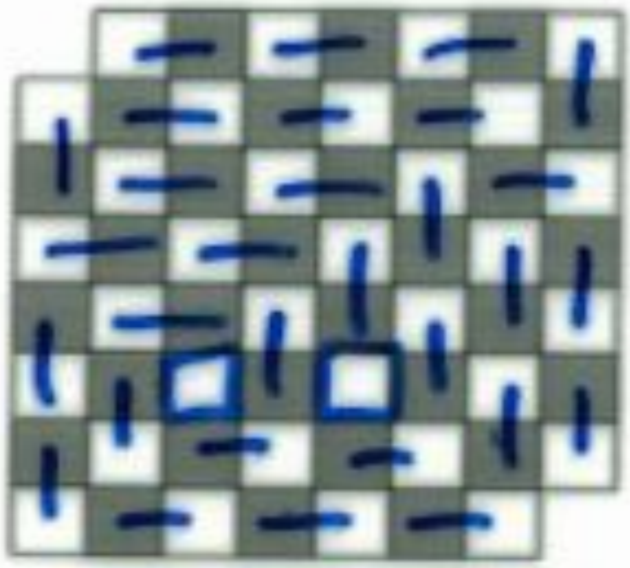
Suppose dominoes cover exactly 2 squares, either vertically or horizontally.

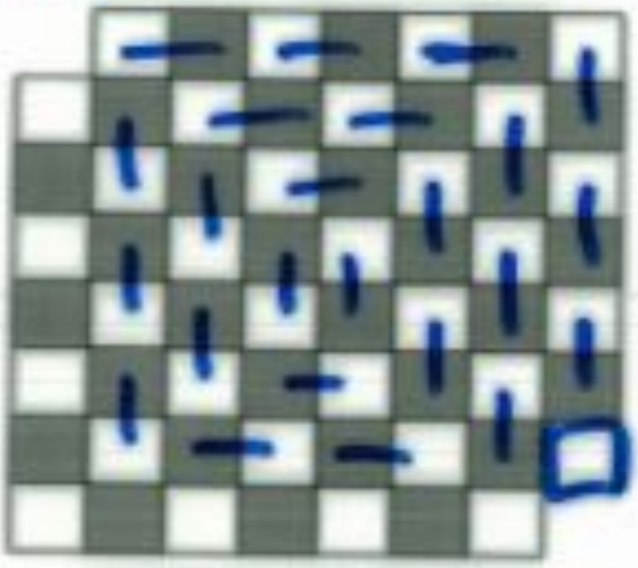
Can you cover all 64 squares using dominoes?

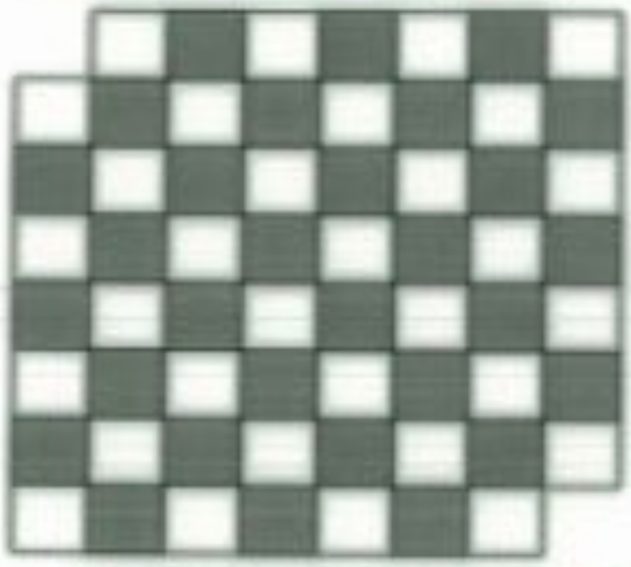
What if we remove two (opposite) corners of the board?

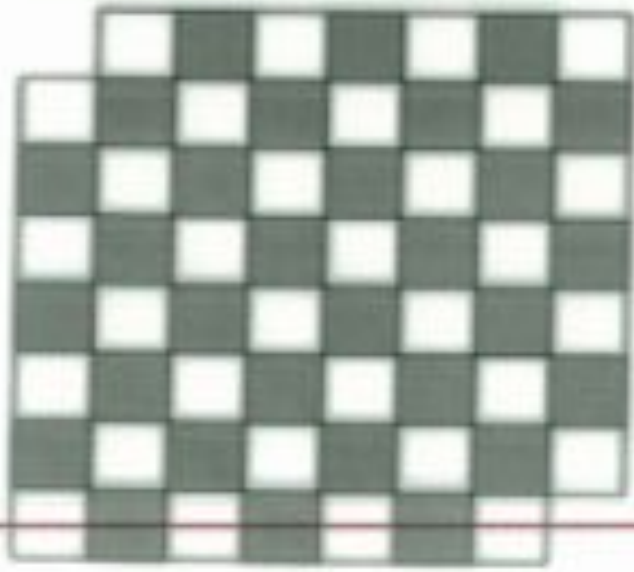
Mutilated Chessboard Problem

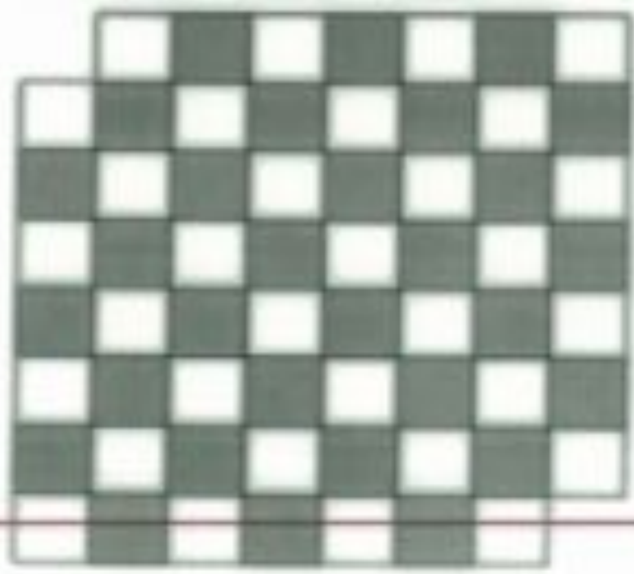












A **mathematical proof** is an airtight logical argument. A correct pf is true for all eternity. (!!)

In symbols, to prove

$p \Rightarrow q$

If p, then q

etc...

we want to create a series of implications w/ statements:

$$p \Rightarrow s_1 \Rightarrow s_2 \Rightarrow s_3 \Rightarrow \cdots \Rightarrow s_n \Rightarrow q$$

KEY: if p is true and each implicatio is true, then q is true as well!

⚠ Basic issue for students in this section: what can you assume is true?

- basic arithmetic
- basic algebra
- also, these def^{n}s:

An integer n is even if $n = 2k$ for some integer k.

n is odd if it has form $n = 2k+1$ for some integer k.

Note these are iff, by convention so: if I know $n = 2(3)+1$, then I can say n is odd.

Also if I know/assume n is odd, $\exists$ integer $k \ni n = 2k+1$.

Ex: Prove: If n is an odd integer, then n^2 is odd as well.

---

p: $[p(n)]$: n is odd

q: $[q(n)]$: n^2 is odd.

Pf: $p(n)$: n is an odd integer

$\Rightarrow s_1$: $\exists\, k$ (integer) $\ni n = 2k+1$.

$\Rightarrow s_2$: $n^2 = (2k+1)^2$

$\Rightarrow s_3$: $n^2 = 4k^2 + 4k + 1$

$\Rightarrow s_4$: $n^2 = 2(\underbrace{2k^2+2k}_{\text{even}})+1$

$\Rightarrow q$: n^2 is odd.

ICK!

In words:

Let n be an odd integer, so $\exists$ integer k s.t. $n = 2k+1$. Then

$$\begin{aligned} n^2 &= (2k+1)^2 \\ &= 4k^2+4k+1 \\ &= 2(2k^2+2k)+1 \end{aligned}$$

which has the form of an odd number. Hence n^2 is odd. ■ ● QED etc.

Proving $p \Rightarrow q$ via $p \Rightarrow s_1 \Rightarrow \cdots \Rightarrow s_n \Rightarrow q$ is called a <u>direct</u> proof. $\exists$ other method

Today/Friday: Pf by contrapositive, contradiction, {cases}

[induction to come later...]

<u>Def</u> The <u>contrapositive</u> of $p \Rightarrow q$ is.

$$\sim q \Rightarrow \sim p.$$

<u>Ex</u> Write contrapositive of:

If $x > 1$, then $x^2 > 1$

$x^2 \leq 1 \Rightarrow x \leq 1.$

If it's raining, the sidewalk is wet.

sidewalk dry $\Rightarrow$ not raining.

Contrapositive is useful b/c of this tautology: $(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$

Proof by Contrapositive: prove $p \Rightarrow q$ indirectly, via a direct proof of $\sim q \Rightarrow \sim p$

Ex: Prove: for an integer n,

n^2 even $\Rightarrow$ n even.

Direct Proof Let n^2 be even,

so $n^2 = 2k$, some k.

then $n = \sqrt{2k} = \sqrt{2} \cdot \sqrt{k}$?!?

Alternatively, we prove the contrapositive, n odd $\Rightarrow n^2$ odd.

we did this in ≈ 3 lines last time.

<u>Proof by</u> <u>Contradiction</u> Let c be a "contradiction" - a stmt which is <u>always</u> false, e.g.: 2 is odd, $0=1$, etc.

<u>Thm</u>: $(\sim p \Rightarrow c) \Leftrightarrow p$

if assuming $\sim p$ leads to total nonsense (a contradiction), then our assumption was wrong, and p is true.

$$[(p \wedge \sim q) \Rightarrow c] \Leftrightarrow (p \Rightarrow q)$$

assume $p \Rightarrow q$ is false, show that it leads to a contradiction. Thus our assumption was wrong, and $p \Rightarrow q$ true.

$\exists$ other tautologies used in pfs - see book - but they're based on same ideas.

<u>Prove</u>: There are infinitely many primes.

<u>Pf</u>: Assume there are finitely many primes, exactly n of them:

$$p_1, p_2, \ldots, p_n$$

Let $a := (p_1 p_2 \cdots p_n) + 1$.

<u>Clearly</u> $a \neq p_i$ for any i, so a is not prime.

Thus a can be factored into primes. But $a \div p_i$ has a remainder of 1, so a cannot be factored.

Hence a is not prime and not composite – a contradiction!

?

Thus our assumption is wrong, and there are infinitely many primes!

Prove $\sqrt{2}$ is irrational

Rephrase If $x^2 = 2$, then x not rational

Pf:

Summary of terms

given $p \Rightarrow q$ implication

$\sim q \Rightarrow \sim p$ contrapositive

$\Big\}$ log. equiv

$q \Rightarrow p$ converse

$\sim p \Rightarrow \sim q$ inverse. contrapos. of converse.

$\Big\}$ log equi

In general $\exists$ <u>no</u> connection between truth values of $p \Rightarrow q$ and its converse.

<u>Ex</u>. • n^2 even $\Leftrightarrow$ n even (T)

n even $\Rightarrow$ n^2 even (T)

• $f(x)$ diff'ble $\Rightarrow$ f continuous. (T

cont $\Rightarrow$ diff'ble. (F

Deductive Reasoning: Showing a conclusi[on] follows from certain premises. ($p \Rightarrow q$)

Inductive Reasoning: Pattern Recogniti[on]

We often use INductive reasoning to find *what* to prove, and DEductive reasoning to prove it!

Last Method: Proof by Cases.

Ex Prove: $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

Pf: Case 1 $a \geq 0,\ b > 0$. $\frac{|a|}{|b|} = \frac{a}{b} \geq 0$

$\vdots$

So $\frac{|a|}{|b|} = \frac{a}{b} = \left|\frac{a}{b}\right|$

Case 4 $a \leq 0,\ b < 0 \ldots$

Last warning: to prove a statement is false, it suffices to give <u>one</u> counter-example:

<u>Ex</u> Give ctr-ex to "For all triangles w/ side lengths a, b, c, $a^2+b^2=c^2$."

But you can't prove a universal statement by checking 1 (or 1,000,000) examples:

<u>Ex</u> For a right $\triangle$ w/ hyp c, we have $a^2+b^2=c^2$?

Have to prove Pyth. Thm

can't just check

3-4-5 $\triangle$.