§7. Functions

Contains a variety of def^{n}'s, including "function" as a rel'n, i.e. subset of a Cartesian Prod.

Most important parts:

- $f: A \to B$ notation, domain, range, codomain.
- injective, surjective, bijective
- images and pre-images of sets, $f(C)$ and $f^{-1}(D)$.
- inverses and compositions.

§7. Functions

$\leq$ Calc I, functions generally have one input $(x, t, \ldots)$ and one output $(y, f(t), g(x))$:

$g(x) = x^2$

$y = \sin t$

$\geq$ Calc III, Linear Algebra we have a more general notation:

$$f: A \to B$$

inputs
domain

outputs
target space
codomain

range f = set of actual outputs

$\{f(a) \mid a \in A\} = \{b \in B \mid b = f(a) \text{ some } a \in A\}$

! f must assign an output to each elt in its domain!

!! In many books B (codomain) called the range.

<u>Ex</u> $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$

$(x,y) \longmapsto \frac{y+1}{x}$ $\quad z = f(x,y)$ $\quad \left[z = \frac{y+1}{x} \right]$

domain: $\{x \neq 0\} \subset \mathbb{R}^2$

$\{(x,y) \in \mathbb{R}^2 \mid x \neq 0\}$

codomain $= \mathbb{R}$, range $f =$

⚠ should write

$f: D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}.$

$D = \{x \neq 0\}$

$f: D \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ $\quad y = f(x) = \sqrt{4 - x^2}$

$x \longmapsto \sqrt{4 - x^2}$

$D = [-2, 2]$

range $f = [0, 2]$

so codomain $\neq$ range.

<u>but</u> I could write/redefine

$f: D \subset \mathbb{R} \to [0, 2]$

$x \longmapsto \sqrt{4 - x^2}$

This book is even more general...

<u>Defⁿ</u> A function f between sets A and B is a nonempty subset* of $A \times B$ s.t. if $(a,b) \in f$ and $(a,b') \in f$, $b = b'$.

* i.e. a relⁿ.

Instead of giving a formula to define f, this method just lists all inputs and their associated outputs.

The extra condition ensures that any input a has just <u>one</u> output.

<u>Ex</u>: $f : \{1,2,3,4\} \longrightarrow \mathbb{N}$ $\quad f(n) = n^2+1$

$f = \{(1,2), (2,5), (3,10), (4,17)\}$

⚠ In this method, a fn f b/w A, B need not have domain $= A$.

$A = \{1, 2, 3\}$, $B = \mathbb{N}$,
$f = \{\{1, 10\}, \{3, 47\}\}$ — $2 \in A$, $2 \notin \operatorname{dom} f$.

If we write $f: A \to B$, it implies $\operatorname{dom} f = A$.

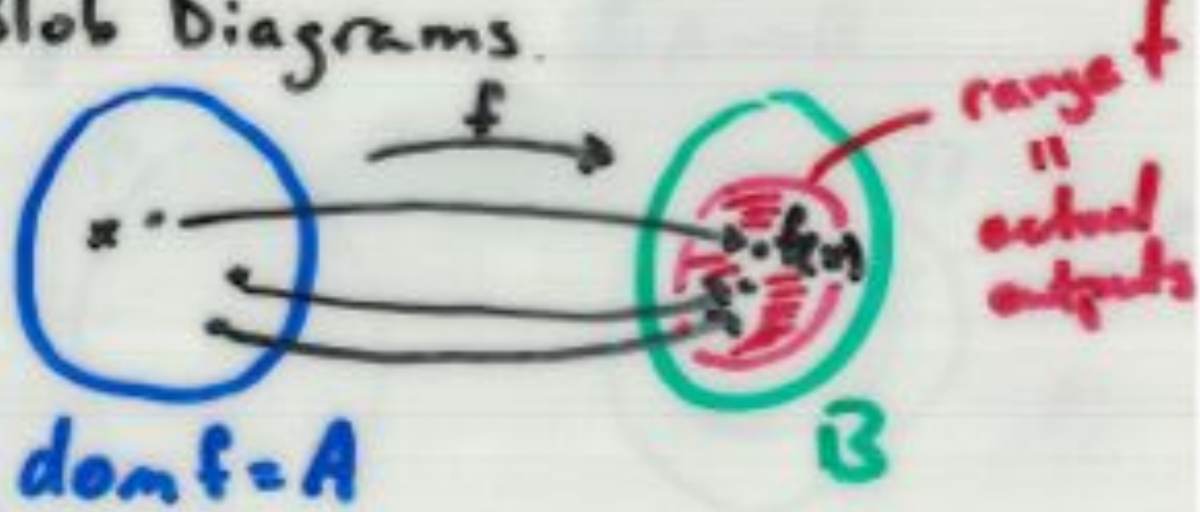
$\exists$ two common ways to visualize fns:

① <u>Graphs</u>: when inputs/outputs are #'s

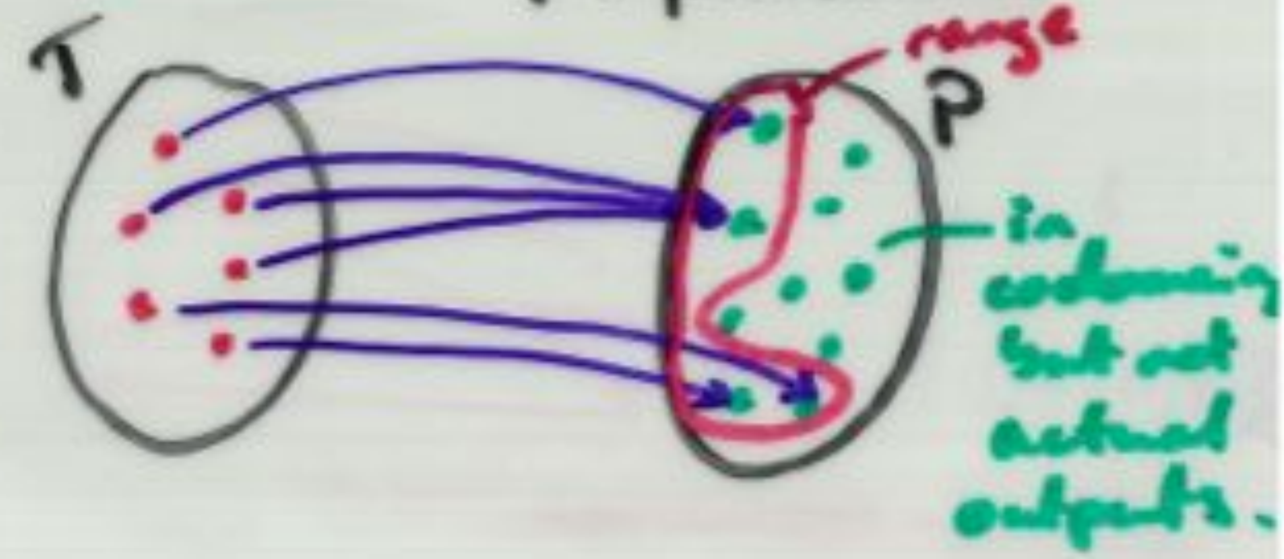
inputs usually horiz. coords
outputs usually vert coords.

$y = f(x)$

② "Blob" Diagrams.

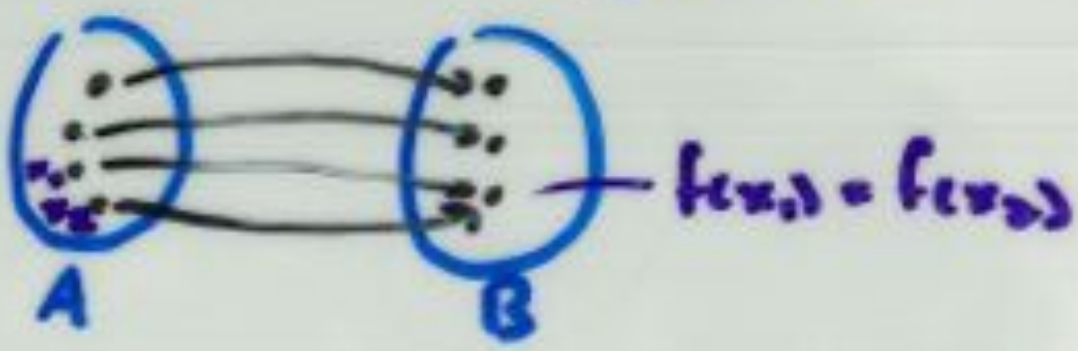


Ex $f: T \to P$ T = tomatos
P = professors.



Glossary of Terms for $f: A \to B$

- **surjective**, **onto**. everything in B gets hit.



$f: A \to B$ is onto if $\forall\, b \in B$
$\exists\, a \in A \ni f(a) = b$.

• **injective**, one to one, 1:1

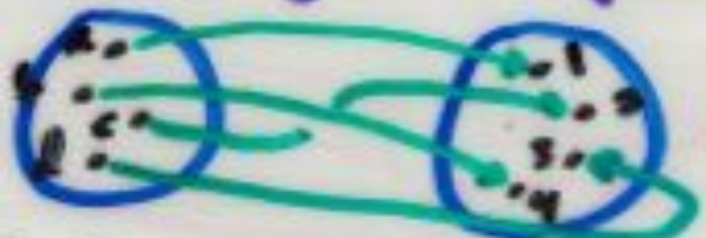
Each output is hit by only one input.

If $f(x_1)=f(x_2)$ then $x_1=x_2$.

OR (CP) $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$.

• **bijective** both injective and surjective.
Each $b \in B$ is hit by exactly one input $a \in A$.

"relabeling"



1:1 correspondence

• **inverse fn of f** is a fn $f^{-1}: B \to A$

s.t. $f^{-1}(f(a)) = a \quad \forall a \in A$.

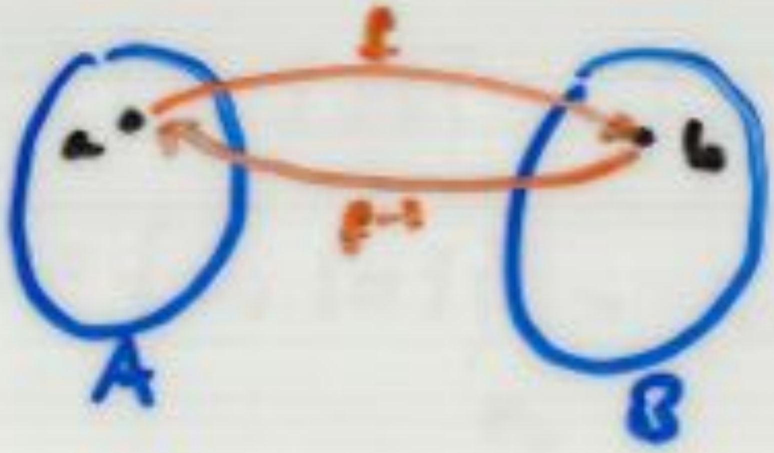
$f(f^{-1}(b)) = b \quad \forall b \in B$.

i.e. $f^{-1} \circ f = id_A : A \to A,\ a \mapsto a$

$f \circ f^{-1} = id_B : B \to B,\ b \mapsto b$

Set def$^{\underline{n}}$ of $f^{-1}: B \to A$

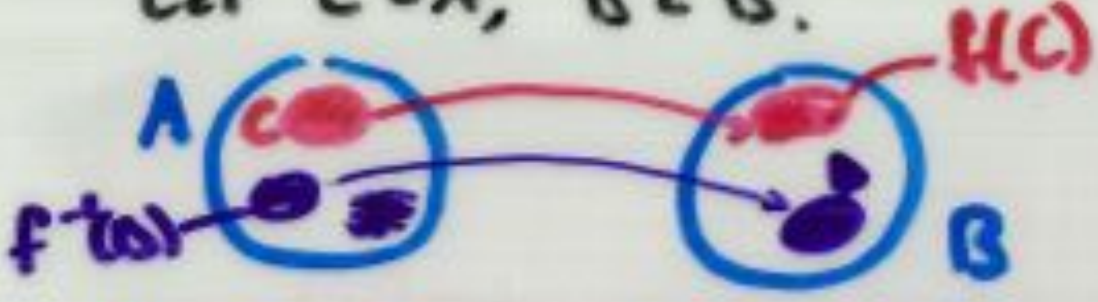
$f^{-1} = \{(b,a) \mid f(a) = b\}$ and if (b,a) and (b,a') then $a = a'$



Existence of f^{-1} depends on f being bijective.
(f is <u>bijection</u>)

• **fns acting on subsets** $f: A \to B$

Let $C \subset A$, $D \subset B$.



$f(C) = \{(x,y) \in f \mid x \in C\}$ (bb)

i.e. $f(C) = \{f(c) \mid c \in C\}$.

"image of C under f".

$f^{-1}(D) = \{a \in A \mid f(a) \in D\}$

or $\{(x,y) \in f \mid y \in D\}$ (bb)

"preimage of D under f."

! doesn't imply f invertible.

Ex $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$.

$f([0,2)) = [0,4)$ $\quad 0 \le x < 2$, $\; ? \le x^2 \le ?$

$f^{-1}((1,4]) = (1,2] \cup [-2,-1)$.

$f(\mathbb{N}) = \{f(n) \mid n \in \mathbb{N}\} = \{1,4,9,16,\dots\}$

$f^{-1}([-9,-4]) = \emptyset$

Composition $g \circ f(x) = g(f(x))$

In pictures, if $f: A \to B$, $g: B \to C$,

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

$g \circ f$. (watch the dir'n!)

READ THE BOOK CAREFULLY AND ASK QUESTIONS. $\exists$ many thms like...

Thm 7.19 (a) f, g surjective $\Rightarrow g \circ f$ surj.

Thm 7.15 (c) $f(C_1 \cap C_2) \subseteq f(C_1) \cap f(C_2)$

Thm 7.17 (a) f inj, $C \subseteq A \Rightarrow f^{-1}[f(C)] = C$

Rather than memorize everything, practice this type of proof so you can "do it on the fly."

Thm 7.19 (a). Let $f: A \to B$, $g: B \to C$ be surjective. Then $g \circ f$ is surj.

$g \circ f$ is surjective.

Note: $g \circ f : A \to C$.

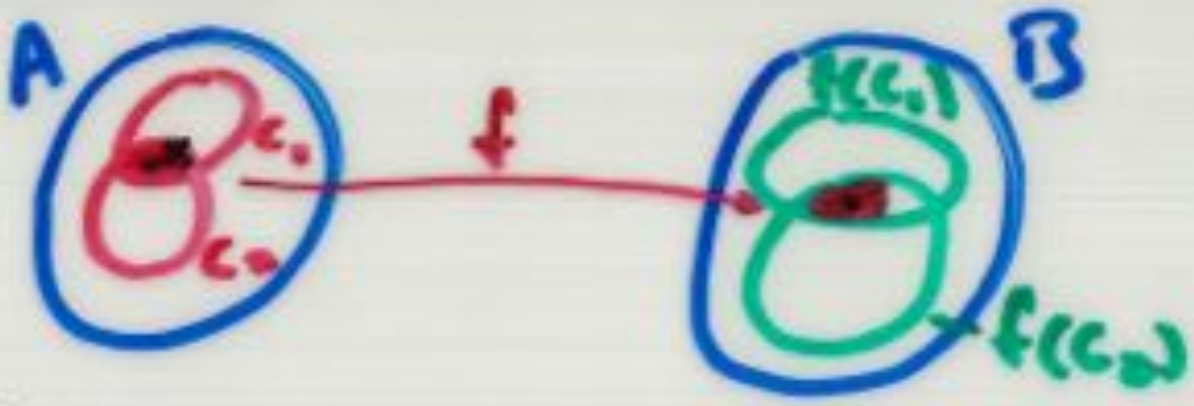
Pf: $g \circ f : A \to C$, so to prove $g \circ f$ is surjective we let $c \in C$, want to show $\exists\ a \in A$ such that $g \circ f(a) = g(f(a)) = c$.

$g: B \to C$ is surjective, so $\exists$ $b \in B \ni g(b) = c$.

$f: A \to B$ is surj., so $\exists$ $a \in A$ s.t. $f(a) = b$.

Then $g \circ f(a) = g(f(a)) = g(b) = c$.

<u>Thm 7.15(c)</u> $f(C_1 \cap C_2) \subseteq f(C_1) \cap f(C_2)$



<u>Pf</u>: Let $x \in C_1 \cap C_2$. Since $x \in C_1$, $f(x) \in f(C_1)$. Since $x \in C_2$, we also have $f(x) \in f(C_2)$. Thus $f(x) \in f(C_1) \cap f(C_2)$.

$\Rightarrow f(C_1 \cap C_2) \subseteq f(C_1) \cap f(C_2)$.