§ 34 Power Series

So far every infinite series has been a sum of preselected, fixed. #'s — given a sequence $(a_n) = (a_1, a_2, a_3, \dots)$ consider the sum $\sum a_n = a_1 + a_2 + \cdots$

In this section, our series are functions which depend on a variable. Two issues:

1. When does this work / make sense
2. Why would we care?!

Def Let $(a_n)_{n=0}^{\infty}$ be a sequence.

$$\sum_{n=0}^{\infty} a_n x^n = a_0 x^0 + a_1 x + a_2 x^2 + \cdots$$
$$= a_0 + a_1 x + a_2 x^2 + \cdots$$

is a power series. a_n is the n^{th} coefficient (i.e. the coeff of x^n).

Notes

① For a specific x, we get a regular old series.

Ex let $a_n = \frac{1}{n+1}$ $\quad \sum_{n=0}^{\infty} \frac{1}{n+1} x^n = 1 + \frac{1}{2}x + \frac{x^2}{3} + \cdots$

$\underline{x=1}$ $\quad \sum_{n=0}^{\infty} \frac{1}{n+1} \cdot 1 = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots$

diverges - harm. series!

$\underline{x = \frac{1}{2}}$ $\quad \sum_{n=0}^{\infty} \frac{1}{n+1} \left(\frac{1}{2}\right)^n = 1 + \frac{1}{4} + \frac{1}{12} + \frac{1}{32} + \cdots$

Converges by...

comp. test: $\frac{1}{n+1}\left(\frac{1}{2}\right)^n \le \left(\frac{1}{2}\right)^n$

ratio test: $\left| \dfrac{\frac{1}{n+2}\left(\frac{1}{2}\right)^{n+1}}{\frac{1}{n+1}\left(\frac{1}{2}\right)^n} \right| = \frac{n+1}{n+2} \cdot \frac{1}{2} \to \frac{1}{2} < 1$

Main Goal for week:

Simultaneously find **all** values of x for which $\sum a_n x^n$ converges.

② Why do all of this?

It gives us another (often useful) way to represent fns.

Ex If $x \in (-1,1)$ $[|x|<1]$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} 1 \cdot x^n = 1 + x + x^2 + x^3 + \cdots$$

nicer than this

Ex $\forall x \in \mathbb{R}$, turns out

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

still nicer than

$\exists$ advantages to power series form (sometimes)

If $e^u = 1 + u + \frac{u^2}{2} + \frac{u^3}{6} + \cdots = \sum \frac{u^n}{n!}$

then set $u = x^2$ to get

$$e^u = e^{x^2} = 1 + (x^2) + \frac{(x^2)^2}{2} + \frac{(x^2)^3}{6} + \cdots$$

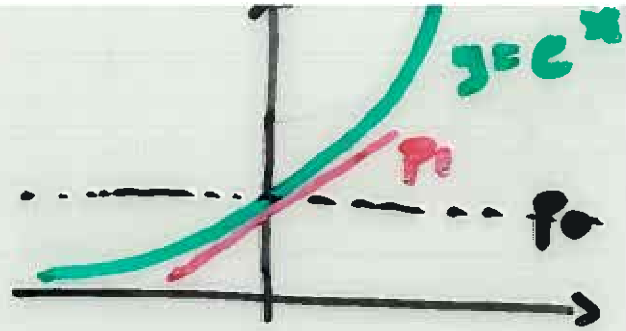
$$= 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \cdots$$

$\int e^{x^2}dx$ cannot be written using "elementary" fns but in Adv Calc / Real Analysis we prove you can integrate a power series term by term:

$$\int e^{x^2}dx = \int \left(1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \cdots\right)dx$$

$$= \left(x + \frac{x^3}{3} + \frac{x^5}{10} + \frac{x^7}{42} + \cdots\right) + C$$

without power series we'd be unable to write down an integral of e^{x^2}.



③ Another way power series arise is through Taylor Polynomials

e^x hard to compute, but polynomials are "easy." (Esp. for a computer.)

Find poly's of degree n which $\approx f(x) = e^x$ near $x = 0$.

↑ match the value and as many derivatives as possible at $x = 0$.

$\underline{n=0}$ $p_0(x) = 1$. $p_0(0) = 1$

$\underline{n=1}$ $p_1(x) = 1 + x$ (tangent line)

$p_1(0) = 1 + 0 = 1$

$p_1'(x) = 1$

$p_1'(0) = 1$.

$\underline{n=2}$ $p_2(x) = 1 + x + \frac{x^2}{2}$

best quadratic appx'n of e^x near $x = 0$.

$p_2(0) = 1$

$p_2'(x) = 1 + \frac{2x}{2}$

$p_2'(0) = 1$

$p_2''(x) = 1$

$p_2''(0) = 1.$

In general $1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!}$ (partial sum of $\sum \frac{x^n}{n!}$) is best n^{th} degree appx'n of e^x near 0.

$$\lim_{n\to\infty}\left(1 + x + \cdots + \frac{x^n}{n!}\right) = e^x$$

i.e. "$p_\infty(x) = e^x$"

Main goal: For which values of x does $\sum a_n x^n$ converge?

Think: find "domain" $f(x) = \sum a_n x^n$

Note: $\sum a_n x^n$ always converges for $\underline{x=0}$: $\sum a_n (0)^n = a_0$

Otherwise use root/ratio tests!

Ex $\sum \left(\frac{x}{2}\right)^n = \sum \left(\frac{1}{2}\right)^n x^n$

For chosen x, apply root test:

$$\lim \left|\frac{x^n}{2^n}\right|^{1/n} = \lim \frac{|x|}{2} = \frac{|x|}{2}$$

Converges if $\frac{|x|}{2} < 1 \iff |x| < 2 \iff x \in (-2, 2)$.

Diverges if $\frac{|x|}{2} > 1 \iff |x| > 2$

Root test gives no info at $\frac{|x|}{2} = 1 \iff |x| = 2 \iff x = \pm 2$.

Check by hand!

Check $\sum \left(\frac{x}{2}\right)^n$ at $x = \pm 2$:

$\underline{x=2}$ $\sum \left(\frac{2}{2}\right)^n = \sum_{n=0}^{\infty} 1^n = +\infty$

$\underline{x=-2}$ $\sum \left(\frac{-2}{2}\right)^n = \sum_{n=0}^{\infty} (-1)^n$ diverges (partial sums oscillate).

$\Rightarrow \sum \left(\frac{x}{2}\right)^n$ converges for $x \in (-2, 2)$.

We say $(-2,2)$ is interval of convergence and radius of conv'ce is $\underline{2}$. (not 4!)

$\underline{\text{Ex}}$ $\sum \underline{\left(\frac{2^n}{n}\right) x^n}$

$$\lim \left| \frac{2^{n+1} \cdot x^{n+1}}{n+1} \cdot \frac{n}{2^n x^n} \right| = \lim \left| \frac{2x \cdot n}{n+1} \right| = |2x| = 2|x|.$$

Converges if $2|x| < 1$
i.e. $|x| < 1/2 \iff x \in (-1/2, 1/2)$.

$\underline{x = 1/2}$ $\sum \left(\frac{2^n}{n}\right)\left(\frac{1}{2}\right)^n = \sum \frac{1}{n}$ diverges.

$\underline{x=-1/2}$ $\sum \left(\frac{2^n}{n}\right)\left(-\frac{1}{2}\right)^n = \sum \frac{(-1)^n}{n}$ conv.

$R = \frac{1}{2}$

IOC: $[-\frac{1}{2}, \frac{1}{2})$

In general . . .

Thm 34.3 Given $\sum a_n x^n$, let $\alpha = \lim |a_n|^{1/n}$.

Let $R = \begin{cases} 1/\alpha, & 0 < \alpha < \infty \\ 0, & \alpha = +\infty \\ \infty, & \alpha = 0 \end{cases}$

Then $\sum a_n x^n$ converges (abs'ly) whenever $|x| < R$, diverges if $|x| > R$.

Notes

① R is radius of convergence.

$|x| < R \iff x \in (-R, R)$

② $R = 0$: converges only at $x = 0$.

③ $R = +\infty$: conv's $\forall x \in \mathbb{R}$.

④ Int. of Conv. (IoC) might be $(-R, R)$, $[-R, R]$, $(-R, R]$, $[-R, R)$

⑤ Prove by applying root test to $\sum (a_n x^n)$

Pf: Given $\sum a_n x^n$, let $b_n = a_n x^n$.

So $\sum a_n x^n = \sum b_n$.

$$\lim |b_n|^{1/n} = \lim |a_n x^n|^{1/n}$$

$$= \lim_{n\to\infty} |a_n|^{1/n} \cdot |x|$$

$$= |x| \cdot \underbrace{\lim |a_n|^{1/n}}_{\alpha} = |x| \cdot \alpha$$

Case $\underline{\alpha = 0}$ $\lim |b_n|^{1/n} = \cdots = |x| \cdot 0 = 0 < 1$ for all x. $\Rightarrow$ Radius of conv'ce is ∞.

$\underline{\alpha = \infty}$ $\lim |b_n|^{1/n} = 0 < 1$ only if $x = 0$ $\Rightarrow R = 0$.

$\underline{0 < \alpha < \infty}$ $\lim |b_n|^{1/n} = |x| \cdot \alpha < 1$

$\Leftrightarrow$ $|x| < \frac{1}{\alpha} = R$.

Similarly...

Thm 34.4 Radius of Convergence R of $\sum a_n x^n$ is $\lim \left| \frac{a_n}{a_{n+1}} \right|$, provided this limit exists.

⚠ Be careful - its $\lim \left| \frac{a_n}{a_{n+1}} \right|$

Sketch of Pf We "know"

$$\lim |a_n|^{1/n} = \lim \left| \frac{a_{n+1}}{a_n} \right|$$

From prev's thm:

$$R = \frac{1}{\alpha} = \frac{1}{\lim |a_n|^{1/n}} = \frac{1}{\lim \left| \frac{a_{n+1}}{a_n} \right|}$$

$$= \lim \left| \frac{a_n}{a_{n+1}} \right|$$ ← num, dem. flipped

⚠ It might be easier to remember $R = \dfrac{1}{\lim \left| \frac{a_{n+1}}{a_n} \right|}$

Find radius, int. of convergence:

(a) $\sum n x^n$ $\quad R = \dfrac{1}{\lim \left|\frac{a_{n+1}}{a_n}\right|} = \lim \left|\dfrac{a_n}{a_{n+1}}\right|$

$a_n = n$

$= \lim \left|\dfrac{n}{n+1}\right| = 1.$

$\underline{x=1}$: $\sum n = +\infty$ $\quad$ $I_{oc} = (-1,1)$

$\underline{x=-1}$: $\sum (-1)^n n$ diverges

(b) $\sum \left(\frac{x}{n}\right)^n = \sum \left(\frac{1}{n}\right)^n x^n$

$R = \dfrac{1}{\lim |a_n|^{1/n}} = \dfrac{1}{\lim \left|\frac{1}{n}\right|} = +\infty$

converges $\forall x \in \mathbb{R}$.

(c) $\sum \left(\frac{1}{n}\right) x^n$ $\quad R = \lim \left|\dfrac{1/n}{1/(n+1)}\right| = \lim \left|\dfrac{n+1}{n}\right|$

$= 1.$ $\quad$ $R = 1$

$\underline{x=1}$: $\sum \frac{1}{n}$ diverges $\quad$ $I_{oc} = [-1,1)$

$\underline{x=-1}$: $\sum (-1)^n \frac{1}{n}$ conv.

(d) $\sum \dfrac{x^n}{n!} = e^x$ $\quad R = +\infty$

Finally: Interval of Conv'ce

may be centered away from origin:

Ex $2 + 2(x-1) + 2(x-1)^2 + \cdots = \sum_{n=0}^{\infty} 2(x-1)^n$

$$= \sum_{n=0}^{\infty} \underbrace{2}_{a} \underbrace{(x-1)}_{r}^{n}$$

$$= \frac{a}{1-r} = \frac{2}{1-(x-1)} = \frac{2}{2-x}$$

iff $|r| = |x-1| < 1$

so interval of convergence

is $x \in (0,2)$,

radius of 1