MATH 3283W: The Completeness Axiom

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How is $\mathbb{R}$ different from $\mathbb{Q}$?

We want to make precise the idea that "$\mathbb{R}$ has no holes", a property that $\mathbb{Q}$ does not enjoy.

Plan:

1. Review the definition of *supremum*.
2. Work with the concept of supremum in examples.
3. State the Completeness Axiom.
4. Prove the Archimedean property.
5. Write down some useful reformulations of the Archimedean property.

Definition of *supremum*

Definition

Let S be a nonempty subset of $\mathbb{R}$. If S is bounded above, then we say that m is a *supremum* of S, and write $m = \sup S$, if:

1. $\forall s \in S, s \leq m$.
2. If $m' < m$, then $\exists s' \in S \ni m' < s'$.

Notes

- Rewrite part 2 as contrapositive.
- Why are we entitled to say "m is **the** supremum of S", if it exists?
- Define *infimum* similarly. We say $l = \inf S$ if ...

Finding suprema and infima and justifying

Example

Let

$$S = \left\{ \frac{n-1}{n+1} : n \in \mathbb{N} \right\}.$$

Find $\sup S$ and $\inf S$.

The Completeness Axiom

The Completeness Axiom

Every nonempty subset S of $\mathbb{R}$ that is bounded above has a supremum (in $\mathbb{R}$).

Notes

- This is a statement about the real numbers that we *accept as true*.
- The hypothesis "nonempty" is necessary. Why?
- The Completeness Axiom is not a property that $\mathbb{Q}$ enjoys.

The Archimedean Property

Theorem

(Archimedean Property) The set $\mathbb{N}$ of natural numbers is unbounded above in $\mathbb{R}$.

Notes

- $\mathbb{N}$ is unbounded above in $\mathbb{R}$?! Of course it is!
- Prove using the Completeness Axiom.

Equivalent forms of the Archimedean Property

Theorem

The following statements are equivalent:

1. *(Archimedean property)* $\mathbb{N}$ *is unbounded above in* $\mathbb{R}$.
2. $\forall z \in \mathbb{R}, \exists n \in \mathbb{N} \ni n > z$.
3. $\forall x > 0, \forall y \in \mathbb{R}, \exists n \in \mathbb{N} \ni nx > y$.
4. $\forall x > 0, \exists n \in \mathbb{N} \ni 0 < 1/n < x$.

Notes

- First let us visualize the meaning of each statement:
  - $3 =$ Tiny steps
  - $4 =$ Capture the flag
- Then, prove $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 1$. (With 4 implications, we actually prove 12.)