§1.1 Logical Connectives

Math consists of statements, sentences which can be classified as true or false – although we might not know which!

Ex Which of these are statements?

p: $2+2=4$ yes (T)

q: $3+3=10$ yes (F)

r: We're in Wisconsin. Yes (F)

s: This is your favorite class.
"Yes", assuming we can rank classes.

t: $x^2 - 4x + 3 = 0$
Yes; truth value depends on x.

u: This statement is false.
<u>No</u> - not a stmt.

v: It's cold outside
Yes, <u>if</u> "cold" is well-defined.

w: Truth is beauty.
Probably not....

Given stmts p, q we can create new ones using logic operators.

→ sentential connectives ←

① Negation $(\neg, \sim)$

$\neg p$ is <u>true</u> when p is false, <u>false</u> when p is true.

We can represent this with "truth table."

p	$\neg p$
T	F
F	T

② Conjunction (and, $\wedge$)

$p \wedge q$ is true when <u>both</u> p, q are true. Otherwise, false.

p	q	$p \wedge q$	$\sim(p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

③ Disjunction (or, $\vee$)

$p \vee q$ true if p is true, or q is true, <u>or</u> both. (else false)

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

⚠ Rarer: exclusive or ($\veebar$, XOR)

p xor q true if exactly one of p, q true.
(else false)

Ex We can combine these operators.

p = Jim is tall
q = Jim has red hair

$p \wedge q$ = Jim is tall and has red hair.

$\sim(p \wedge q)$ = not (Jim is tall and has red hair)

= Jim is (not tall) or (does not have red hair)

don't write $\sim p \wedge q$

i.e. $(\sim p) \wedge q$

You try: truth table for $(\sim p) \vee (\sim q)$

p	q	$\sim p$	$\sim q$	$(\sim p) \vee (\sim q)$	$p \wedge q$	$\sim(p \wedge q)$
T	T	F	F	F	T	F
T	F	F	T	T	F	T
F	T	T	F	T	F	T
F	F	T	T	T	F	T

⚠ $\sim(p \wedge q)$ is T/F precisely when $(\sim p) \vee (\sim q)$ is T/F. We say these stmts are <u>logically equivalent</u>, $\Leftrightarrow$

You just proved one of De Morgan's Laws:

$$\sim(p \wedge q) \iff (\sim p) \vee (\sim q)$$

$$\sim(p \vee q) \iff (\sim p) \wedge (\sim q)$$

4 Implications ($\Rightarrow$, if.... then....)

If p, then q. ($p \Rightarrow q$)

p = antecedent (hypothesis)

q = consequent (conclusion)

We use the convention that $p \Rightarrow q$ false only if p is true, q is false.

$$(p \Rightarrow q) \iff \sim(p \wedge (\sim q))$$

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Ex Determine truth values:

(T) If $\underbrace{\text{2 is positive}}_{T}$, then $\underbrace{\text{4 is even}}_{T}$.

(F) If $\underbrace{\text{3 is odd}}_{T}$, then $\underbrace{\text{pigs can fly}}_{F}$.

(T) If $\underbrace{\text{pigs can fly}}_{F}$, then $\underbrace{\text{I'm a rock star}}_{T/F}$.

If $p \Rightarrow q$ is true and $q \Rightarrow p$ is true, we say $p \Leftrightarrow q$ is true. This is shorthand for logically equivalent, "p if and only q"
"p iff q".

p	q	$p \Rightarrow q$	$q \Rightarrow p$	$p \Leftrightarrow q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

⚠ $p \Leftrightarrow q$ is called a "biconditional statement" and could be true or false

p: n is even
q: n is odd

"$p \Leftrightarrow q$" is false.

In math we're more interested in biconditionals which are tautologies

stmt which is always true.

$$\sim(p \Rightarrow q) \Longleftrightarrow p \wedge (\sim q)$$

Think: rewriting stmt in equiv. way.