# F'11 Exam 1: Consider this relation on $\mathbb{R}^2$:

$$(a,b)\,R\,(c,d) \text{ iff } a^2+b^2=c^2+d^2$$

## (a) Prove $R$ is an equivalence relation

**Reflexive** Let $(a,b)\in\mathbb{R}^2$. Then $a^2+b^2 = a^2+b^2$, so $(a,b)R(a,b)$

**Symmetric** Let $(a,b)$ and $(c,d)$ be in $\mathbb{R}^2$ such that $(a,b)R(c,d)$. Then $a^2+b^2=c^2+d^2$.
Thus $c^2+d^2=a^2+b^2$, so $(c,d)R(a,b)$.

**Transitive** Suppose $(a,b)R(c,d)$ and $(c,d)R(e,f)$.
Then $a^2+b^2=c^2+d^2$
and $c^2+d^2=e^2+f^2$
Because equality of real #'s is trans've,
$a^2+b^2=e^2+f^2$, so $(a,b)R(e,f)$.

Equivalence Rel'ns are our way of Generalizing "equality" to other contexts, like Geometry.

Given an eq. rel'n, an important question is:

For a certain x, what is x equivalent to?

<u>Def</u> Given an eq. rel'n $R$ on a set $S$, the **equivalence class** of $x \in S$ is:

$$E_x = \{ y \mid yRx \}$$

Again: $E_x = \{y \in S \mid xRy\}$



**Ex** Clock Arithmetic

$E_{12} = \{\ldots, -12, 0, 12, 24, 36, \ldots\}$

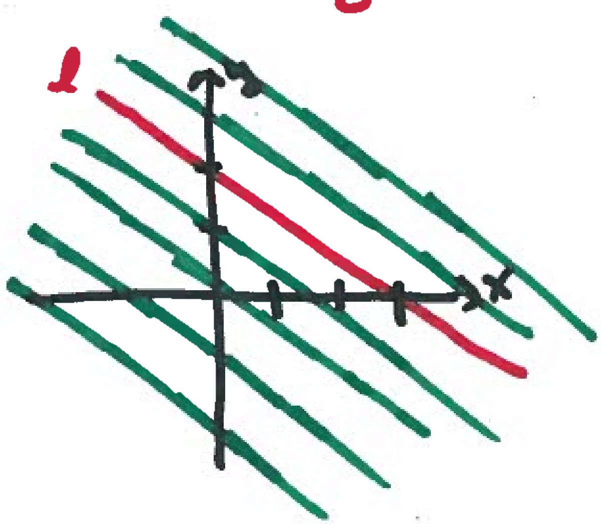
$E_{11} = \{\ldots, -13, -1, 11, 23, 35, \ldots\}$

$\vdots$

$E_1 = \{\ldots, -23, -11, 1, 13, 25, \ldots\}$

**Ex** Lines, $\parallel$.

$\ell: 2x + 3y = 6$




$E_\ell$ = all lines parallel to $\ell$.

= all lines w/ slope $-\frac{2}{3}$

$= \{2x + 3y = c \mid c \in \mathbb{R}\}$

$= \{y = -\frac{2}{3}x + b \mid b \in \mathbb{R}\}$

$(a,b) R (c,d)$ iff $a^2+b^2 = c^2+d^2$

(b) Describe the equivalence class of $(3,4)$. What does it look like geometrically?

$$E_{(3,4)} = \{ (c,d) \mid (c,d) R (3,4) \}$$

$$= \{ (c,d) \mid c^2+d^2 = 3^2+4^2 \}$$

$$= \{ (c,d) \mid \underbrace{c^2+d^2 = 5^2}_{x^2+y^2=25} \}$$

A circle centered at $(0,0)$ of radius 5.

# More abstract example (like 2.2 #30)

Let $S = \{a, b, c, d\}$. What is the eq rel'n $R$ on $S$ with the <u>fewest members</u> such that $(a,b), (c,b) \in R$?

Remember, $R \subset S \times S$, so $(a,b) \in R$ means $aRb$.

We need $R$ to be <u>reflexive</u>: $aRa$, $bRb$, $cRc$ and $dRd$.

We need $R$ to be <u>symmetric</u>: $aRb \Rightarrow bRa$, $cRb \Rightarrow bRc$.

——— " ——— <u>transitive</u>: $aRb, bRc \Rightarrow aRc$

$$R = \{(a,a), (b,b), (c,c), (d,d), \underbrace{(a,b), (c,b)}_{}, (b,a), (b,c)$$

(given) (symmetry)

(trans'ty)

$$(a,c), (c,a)\}$$

& also for symmetry

(has 10 of 16 possible ordered pairs)

$R = \{ (a,a), (b,b), (c,c), (d,d), (a,b), (c,b), (b,a), (b,c),$
$(a,c), (c,a) \}$

## Equivalence Classes

$E_a = \{a, b, c\} = E_b = E_c$

$E_d = \{d\}$

So $R$ "partitions" $S$ into two sets.

**Proposition** Different Eq. Classes are disjoint.

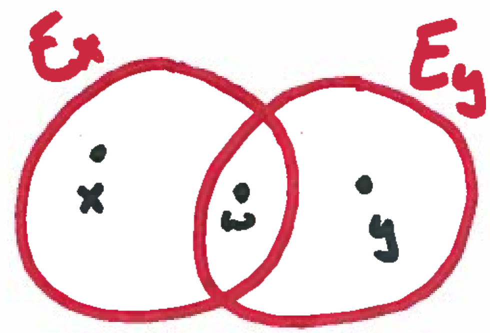
If $E_x$ and $E_y$ are equiv. classes, then $E_x \cap E_y = \emptyset$ or $E_x = E_y$.

Pf Let $x, y \in S$ with eq rel'n $\sim$.

If $E_x \cap E_y = \emptyset$, we're done!

Otherswise $\exists\ w \in E_x \cap E_y$.

We can prove $E_x \subset E_y$ and $E_y \subset E_x$, which means $E_x = E_y$.




Suppose $z \in E_x$. Then $z \sim x \sim w \sim y$, so by transitivity, $z \sim y$ and thus $z \in E_y$. $\Rightarrow E_x \subset E_y$

Other dir'n similar.

**Def** A partition of a set $S$ is a collection $\mathcal{P}$ of non-empty subsets of $S$ such that


(a) $\forall x \in S \ \exists A \in \mathcal{P}$ s.t. $x \in A$

(b) $\forall A, B \in \mathcal{P}$ either $A \cap B = \emptyset$ or $A = B$.

**Ex** $S = \{\underline{1}, \underline{2}, \underline{3}, \underline{4}, \underline{5}, \underline{6}, 7, \underline{8}, 9, \underline{10}\}$

$A = \{1, 2, 3\}$ $B = \{4, 6, 8, 10\}$ $C = \{5\}$, $D = \{7, 9\}$

$\mathcal{P} = \{A, B, C, D\}$ is a part'n of $S$.

**Ex** Assigning $6^{th}$ Graders to soccer teams.

(a) means every kid is on a team.

(b) means each child on just <u>one</u> team.

# KEY IDEA: Equivalence Classes form a partition

Any elt in a set $S$ with eq. rel'n $\sim$ belongs to an eq class. (Ex), and the classes are disjoint.

**Ex** $S$ = UMN students, $xRy \Leftrightarrow x,y$ born in same year

R is an eq. rel'n (you check!)

$$\vdots$$
$$E_{1992} = \{ \text{———— " —— '92} \}$$
$$E_{1993} = \{ \text{people born in '93} \}$$
$$E_{1994} = \{ \text{—— " —— 94} \}$$
$$\vdots$$

$P = \{ E_{1900}, \ldots, E_{2013} \}$ forms a partition of student body.

## Thm 2.2.17

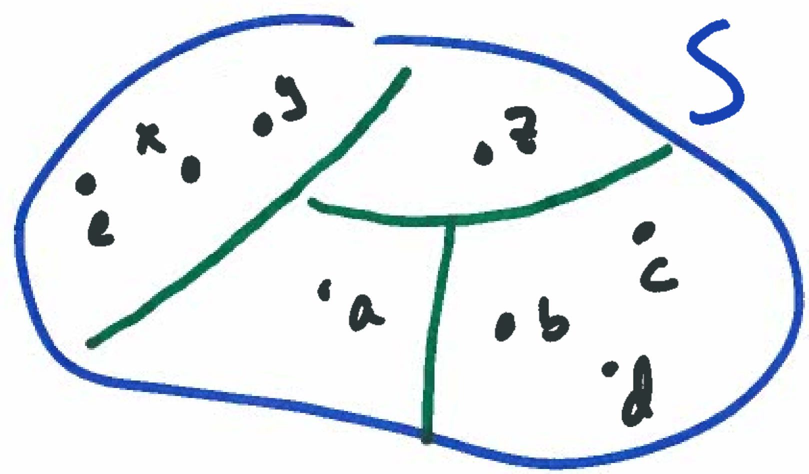
(a) If S has eq rel'n R, the eq. classes form a partition of S.

(b) If $\mathcal{P}$ is a partition of S, the rel'n

$$xRy \iff x, y \text{ are in same set of the part'n}$$

is an eq. rel'n.

Ex




$x \sim y \sim e, \quad a \sim a, \quad z \sim z, \quad b \sim c \sim d$