Math Fun Fact Ham Sandwich Thm

Take two pieces of bread and a slab of ham and place them anywhere in the universe. (!!)

With one slice of a (very long...) knife you can simultaneously cut all three in half!

§2.2 Relations

Or: "Sophisticated def$^{\underline{s}}$ of things you (mostly) already know"

$\exists$ five important def$^{\underline{s}}$ in §2.2:

1. Ordered Pair
2. ☆ Cartesian Product
3. ☆ Relation
4. ☆☆ Equivalence Relation
5. (partition into) equiv. classes.

Sets are unordered:

But often order matters - with pts/vectors,

$(1,2) \neq (2,1)$ $\langle 3,4,5 \rangle \neq \langle 5,4,3 \rangle$

Option 1 Define a new "ordered set."

Option 2 Mathematicians like building everything out of a few basic objects.

Def The ordered pair (a,b) is the set

$$(a,b) = \{\{a\},\{a,b\}\} = \{\{a,b\},\{a\}\} = \{\{b,a\},\{a\}\}$$

Ex

$$(1,2) = \{\{1\},\{1,2\}\}$$
$$\quad\ \neq \qquad\quad =$$
$$(2,1) = \{\{2\},\{2,1\}\}$$

not equal, as desired.

Thm $(a,b) = (c,d) \iff a = c$ and $b = d$.

Pf ($\Leftarrow$) Suppose $a = c$ and $b = d$. By def'n,

$$\begin{aligned}(a,b) &= \{\{a\},\{a,b\}\}\\ &= \{\{c\},\{c,d\}\}\\ &= (c,d).\end{aligned}$$

(You try: $\Rightarrow$)

Def Cartesian Product of sets A, B is

$$A \times B = \{(a,b) \mid a \in A,\ b \in B\}$$

i.e. the set of all ordered pairs where 1st coord is from A, 2nd from B.

Watch out! $A \times B \neq B \times A$ in gen'l.

Examples Pts/vectors (x,y) live in

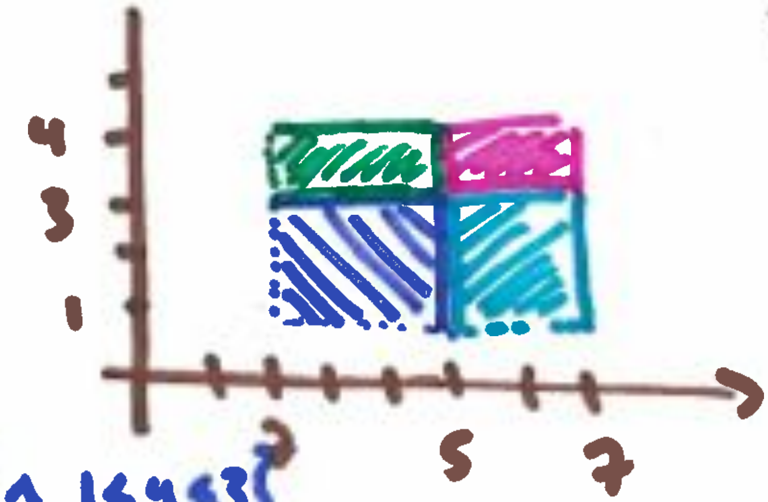
$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x,y) \mid x \in \mathbb{R},\ y \in \mathbb{R}\}.$$

$$\mathbb{R}^3 = (\mathbb{R} \times \mathbb{R}) \times \mathbb{R} = \{((x,y), z) \mid (x,y) \in \mathbb{R}^2,\ z \in \mathbb{R}\}$$

$$\text{"="} \{(x,y,z) \mid x,y,z \in \mathbb{R}\}.$$

Ex $A=(2,5]$ $B=(1,3]$

$C=[5,7]$ $D=[3,4]$



$A\times B=(2,5]\times(1,3]=\{(x,y)\mid 2<x\le 5 \wedge 1<y\le 3\}$

$A\times D=\{(x,y)\mid 2<x\le 5 \text{ and } 3\le y\le 4\}$

$C\times B=\{(x,y)\mid 5\le x\le 7 \text{ and } 1\le y\le 3\}$

$C\times D=$

Relations often we're interested in the relationships b/w elts of sets:

With #'s: $a<b$, $x \geq y$.

With shapes: $\triangle ABC \cong \triangle DEF$ (or similar, or same area...)

Technical Def A *relation* between A, B is a subset $R \subset A \times B$. If $(a,b) \in R$ we write aRb and say "a is related to b".

Notes ① If $A=B$, we say R is a relation on A.

② $aRb \Rightarrow (a,b) \in R \subset A \times B \neq B \times A$ and b not nec. related to a!

Ex $P = \{A, B, C, D\}$

$S = \{\text{house, car, cat, dog, bike, grapes}\}$

A owns house
B owns house $\Big\}$ A, B married.

C owns cat, dog, grapes

"owns" is a rel'n. As a subset of $P \times S$,

owns $= \{$(A, house), (B, house), (C, cat), (C, dog), (C, grapes)$\}$.

Notice: D owns nothing. Also, nobody owns car, bike.

Ex $=$ is a relation on $\mathbb{Z}$, represented by

$\{, \ldots, (-2,-2), (-1,-1), (0,0), (1,1), (2,2), \ldots\} \subset \mathbb{Z} \times \mathbb{Z}$

Ex $<$ is a rel'n on $\mathbb{R}$, repr'd by

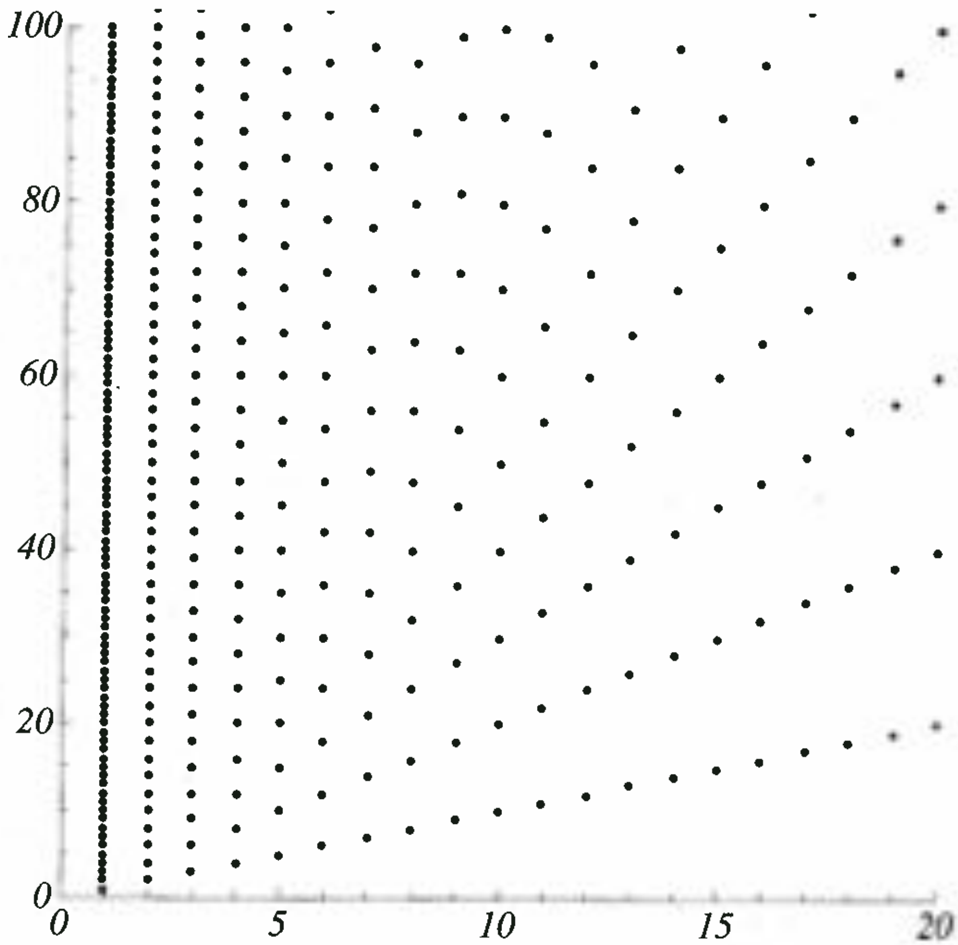
$\{(x,y) \mid x < y\} \subset \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$

Similarly, $x \leq y$ repr'd by

Ex Rel'n on $\mathbb{N}$: "divides," written $|$.

$a|b$ means a divides into b w/ R=0

as a set, $\{(1,1), (1,2), (1,3), \ldots\ldots,$
$(2,2), (2,4), \ldots$
$\vdots \}$



Ex Clock arithmetic, rel'n on $\mathbb{Z}$

informally: every time we hit 12, we wrap around back to 0.

$a \equiv b$ or $a = b \bmod 12$ if:

(1) a and b have same remainder when divided by 12.

(2) OR, $a = b + 12k$ for some $k \in \mathbb{Z}$.

Def An <u>equivalence</u> <u>relation</u> R is a rel'n on a set S which satisfies these conditions

$\forall\, x, y, z \in S$: (1) Reflexive: xRx

(2) Symmetric: $xRy \Rightarrow yRx$

(3) Transitive: $xRy \wedge yRz \Rightarrow xRz$

Ex Which of these are equiv. rel'ns?

- $\mathbb{N}, <$ <u>No</u>! Not reflexive ($3 \not< 3$), not symm. <u>Is</u> trans.
- $\mathbb{N}, \leq$ <u>No</u>! Refl, trans, <u>not</u> symmetric.
- $\mathbb{N}, =$ Yes!
- $\mathbb{N}, \mid$ (divides) <u>No</u>! (Not symm)

- Polygons, $\cong$ (congruent) Yes!
- Polygons, $\sim$ (similar) Yes!
- Lines, $\parallel$ Yes!
- Lines, $\perp$ No! (only symm)

- $\mathbb{Z}$, $\equiv$ ("mod 12" clock arithmetic)

Recall $a \equiv b$ means $a = b + 12k$, some $k \in \mathbb{Z}$.

Check the 3 conditions:

(1) Reflexive. Let $a \in \mathbb{Z}$. Then $a \equiv a$ because $a = a + 12(0)$. (i.e. set $k = 0$).

(2) Symmetric. Let $a, b \in \mathbb{Z}$ with $a \equiv b$, i.e. $a = b + 12k$, some k. Then $b = a + 12(-k)$, so $b \equiv a$.

(3) Transitive. (You finish)