§2.3 Functions

This section contains <u>many</u> def$\underline{^{n}}$'s, including "function" as a relation, i.e. subset of Cart. Product.

We'll cover this section in depth. You will make your future mathematical lives <u>much</u> easier if you put in the effort to learn:

1. $f: A \to B$ notation, domain, codomain, range
2. injective, surjective, bijective (onto, 1:1)
3. fn inverses, preimages
4. compositions.

Algebra Through Calculus A function is a formula or rule which takes each input and transforms it to an output. inputs x, t, θ

outputs $f(x), g(t) = y$

MV Calc/Lin. Alg./Etc We use a more general notation.

$$f: A \to B$$

inputs
domain

(potential)
outputs
target space
codomain

range(f) = set of actual outputs $= \{f(a) \mid a \in A\}$
$= \{b \in B \mid \exists a \ni f(a) = b\}$

⚠ f must assign exactly one output to each element in domain.

⚠!! In many books, range = potential outputs, image = actual outputs

Ex $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ i.e. $f(x,y) = x^2y^2$

$(x,y) \longmapsto x^2y^2$

domain $= \mathbb{R}^2$

codomain $= \mathbb{R}$

range $= \mathbb{R}^+ = \{x \geq 0\}$

This book is even more general - at least, at first....

Def A function between sets A and B is a non-empty subset of $A \times B$ (i.e. a relation) such that if (a,b) and $(a,b') \in f$ then $b = b'$. ⊛

Instead of giving formula or rule, this method lists all inputs with their corresponding outputs.

⊛ means each input has just one output.

Ex $f: \mathbb{N} \to \mathbb{N}$, $f(n) = n+1$ becomes

$$f = \{(1,2), (2,3), (3,4), \ldots\}$$

$g: \mathbb{R} \to \mathbb{R}$, $g(x) = \sin x$ becomes

$$g \neq \{(1, \sin 1), (0, \sin 0), (\tfrac{1}{2}, \sin \tfrac{1}{2}), \ldots\}$$ (Can't "list out" $\mathbb{R}$)

$$g = \{(x, \sin x) \mid x \in \mathbb{R}\}$$

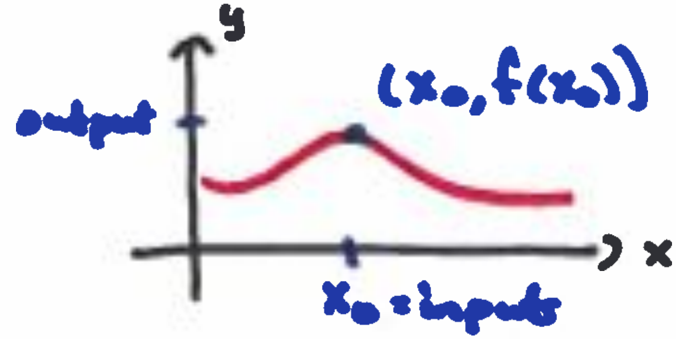
⚠ In this defn of $f \subseteq A \times B$, dom f need not be A!

Ex $A = \{2,3,4\}$ $B = \{$△, □, ⬠$\}$

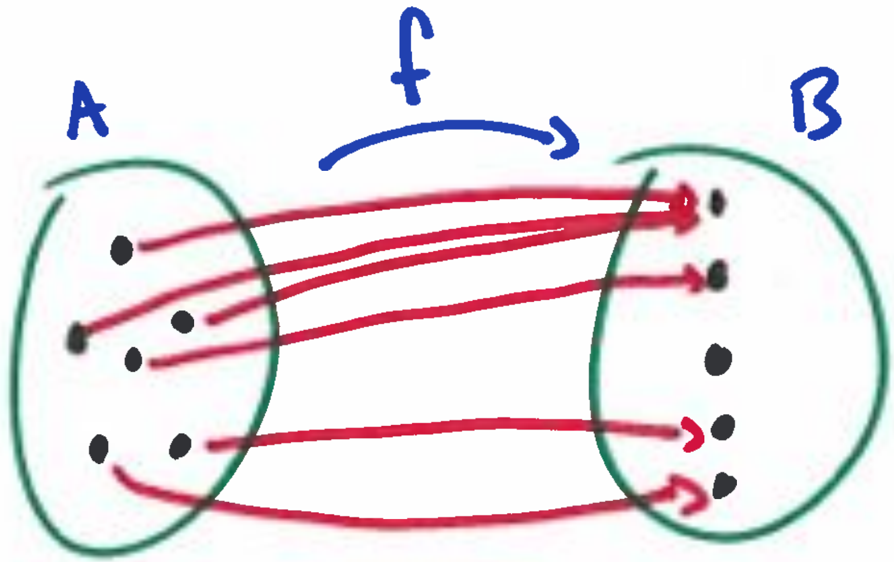
$f = \{(3,$ △$), (4,$ □$)\}$ Here $2 \in A$, but $2 \notin \operatorname{dom} f = \{3,4\}$

Usually we stick with $f: A \to B$, read "f is a fn from A to B". In this not'n, dom $f = A$.

You're used to graphing fns:



We can also visualize them as "generic blobs."



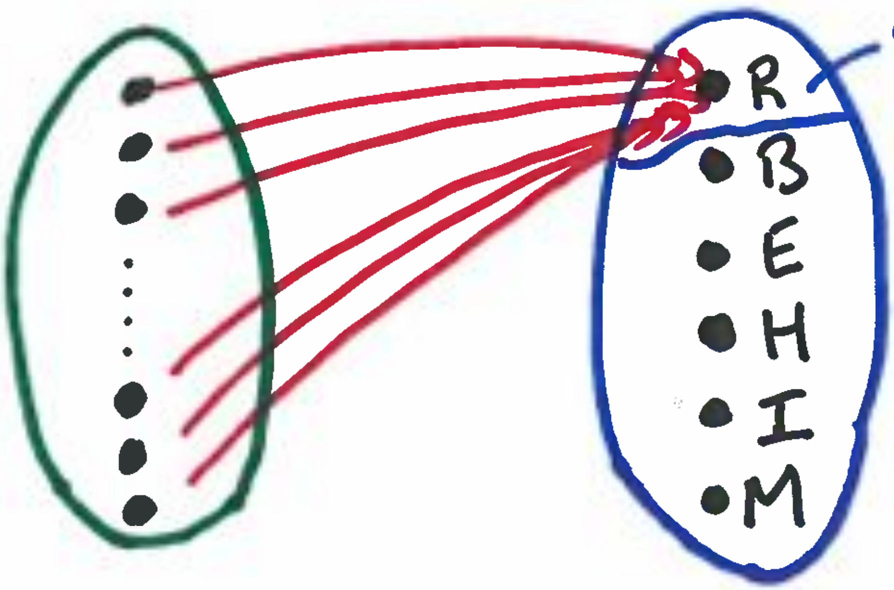
Example $f: X \to Y$

X = set of 3283W students

Y = {Rogness, Baker, Ewing, Honda, Ismail, Morawski}

$f(x)$ = which instructor is hit by a tomato thrown by student x.

range $(f) = \{R\} \neq Y$



$f(x)$ = the person who writes exams

not surjective

not injective (178:1)

Example $f: X \to Y$

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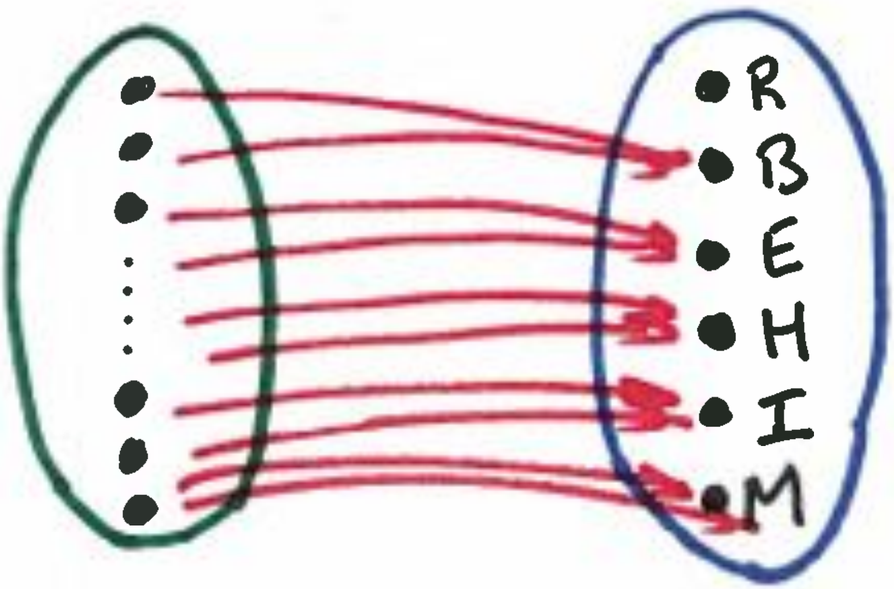

$f(x)$ = person grading your HW....

not surj, not injective

Example $f: X \to Y$

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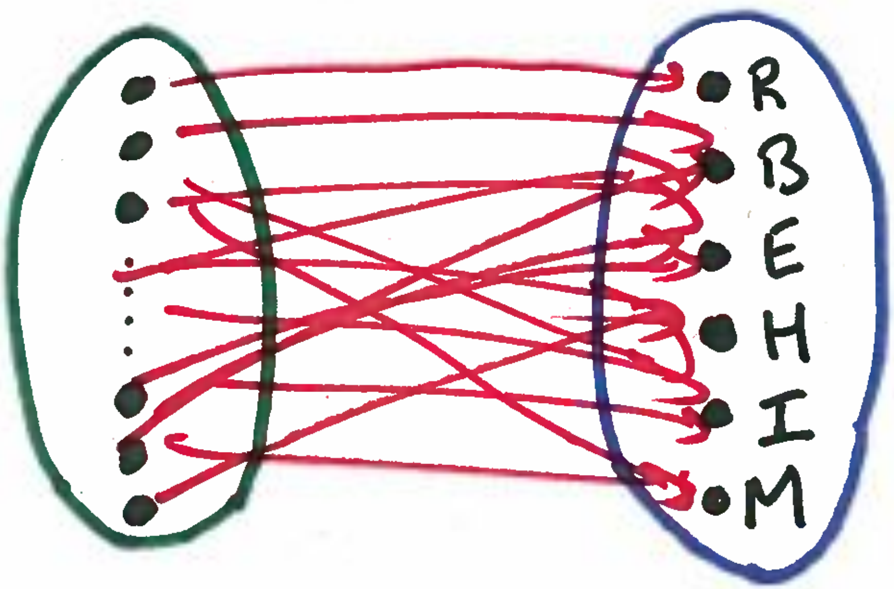
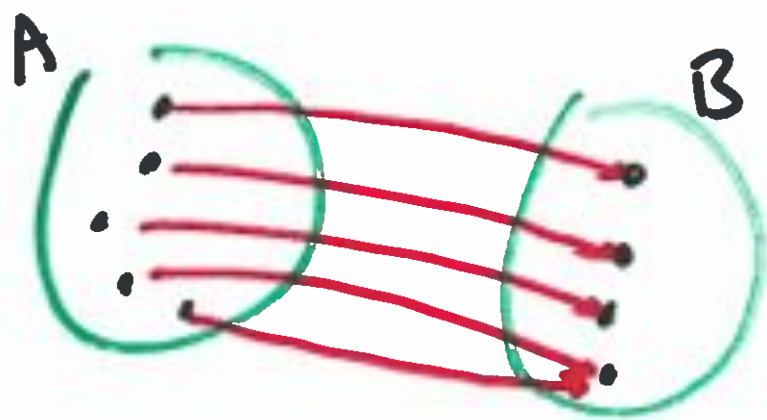


1st six students ensure everybody is hit; random after that.

surjective, not injective

Hugely important properties of $f: A \to B$

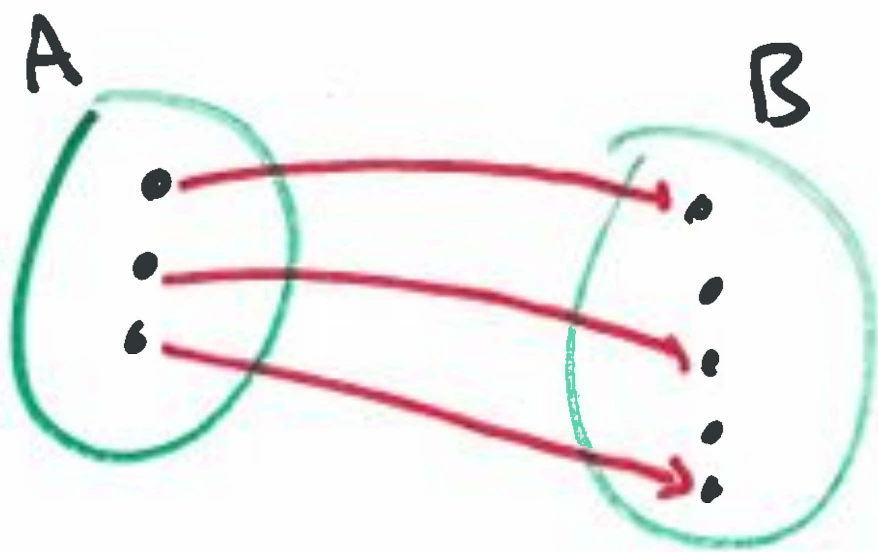
Def f is surjective (onto, is a surjection) if every potential output actually is an output, i.e. $\text{range}(f) = B$.



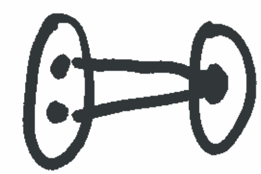
Relation Version: $\forall\, b \in B\ \exists\, a \in A \ni (a,b) \in f$.

Function Notation: $\forall\, b \in B\ \exists\, a \in A \ni f(a) = b$.

Def f is injective (one to one, 1:1, an injection) if no two elts are sent to same output.



Not



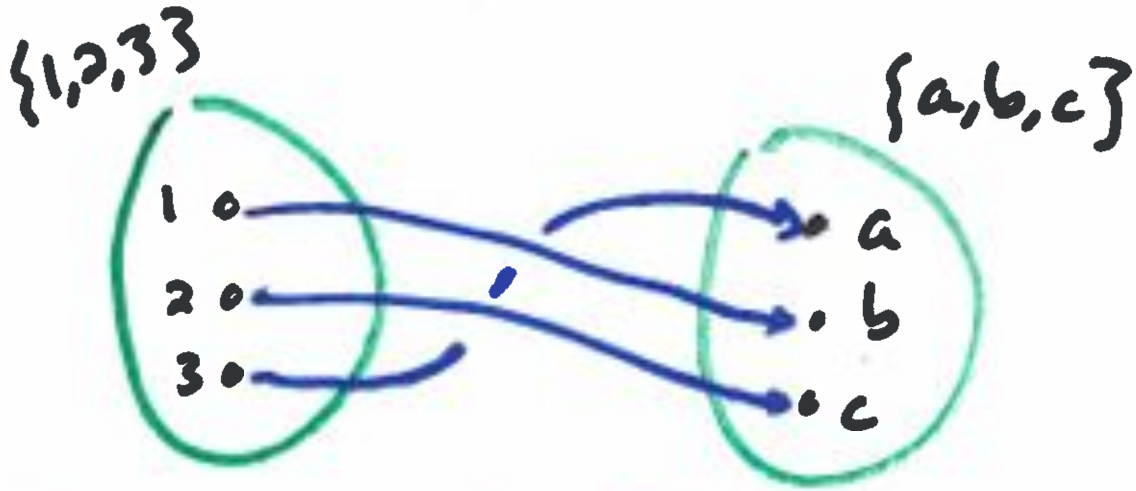
Relation Version If $(a,b) \in f$ and $(a',b) \in f$ then $a = a'$

Function Notation If $f(a) = b$ and $f(a') = b$ then $a = a'$.

OR if $a \neq a'$ then $f(a) \neq f(a')$

(i.e the contrapositive)

Def f is a bijection (is bijective, is a 1:1 correspondi
if it is injective and surjective.



If $\exists$ bijection $A \to B$ it means A, B are "equiv" (see §2.4) – essentially the same sets with elements relabeled.

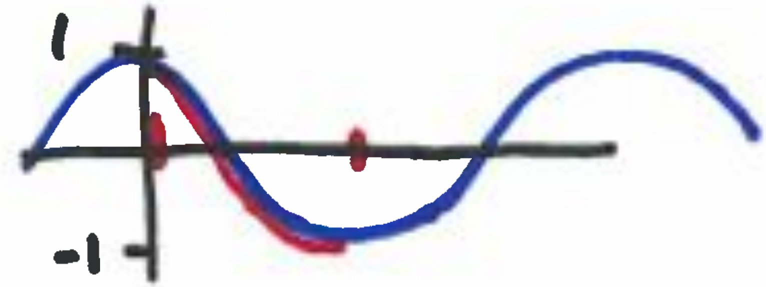
"Redefining Fns"

There is no such thing as a fn which is not surjective – just poorly defined[1] fns....

[1] described

-(My undergraduate adviser)

Ex $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \cos x$.



Not surjective b/c
$\text{range}(f) = [-1,1] \neq \mathbb{R}$

But if I redefine $f: \mathbb{R} \to [-1,1]$, $f(x) = \cos x$
f is now surjective without changing domain, any fn values, etc.

Not injective b/c $\cos(0) = \cos(2\pi) = 1$. If I restrict domain to $f: [0,\pi] \to [-1,1]$, it's injective – BUT a "new" fn w/

Recall composition notation:

Thm 2.3.20 Let $f: A \to B$ and $g: B \to C$.

(a) f, g surjective $\Rightarrow$ $g \circ f$ surjective

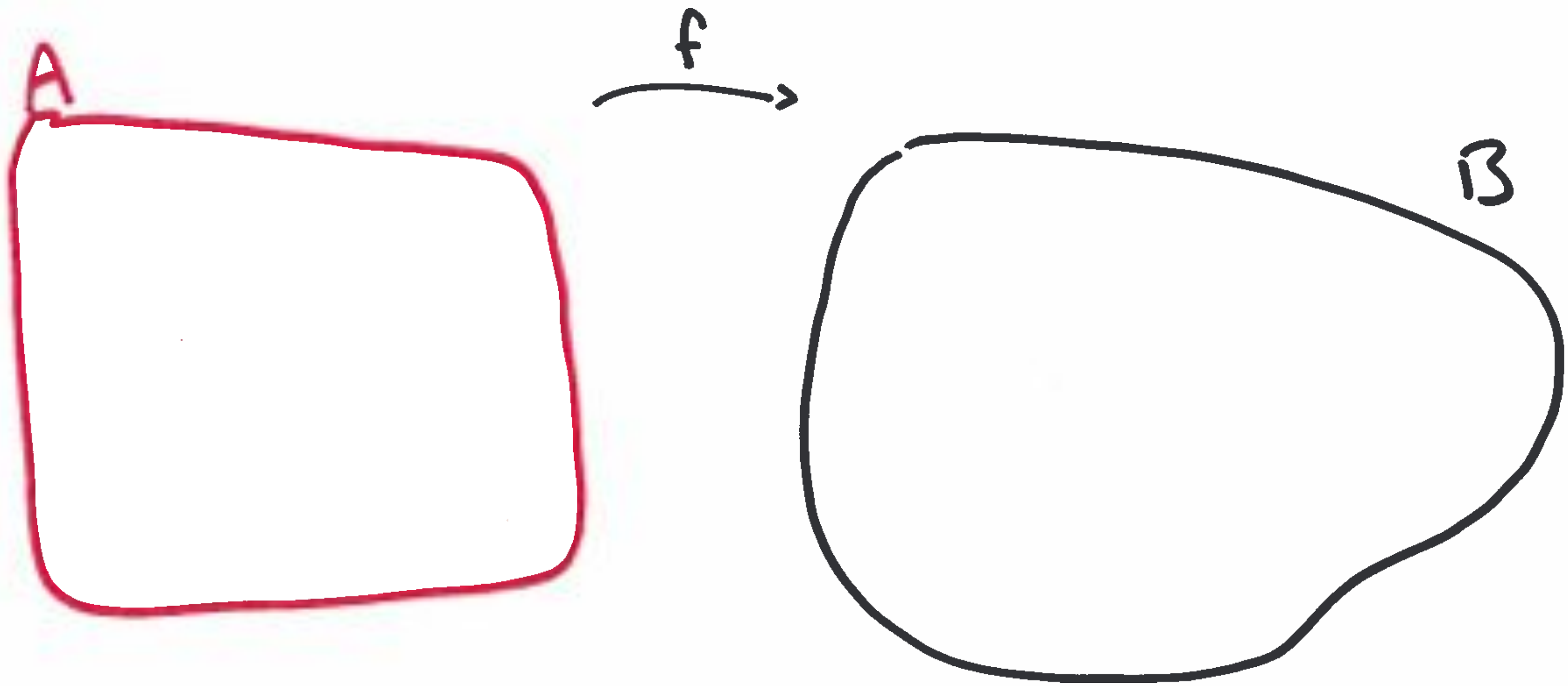
(b) f, g injective $\Rightarrow$ $g \circ f$ injective

(c)

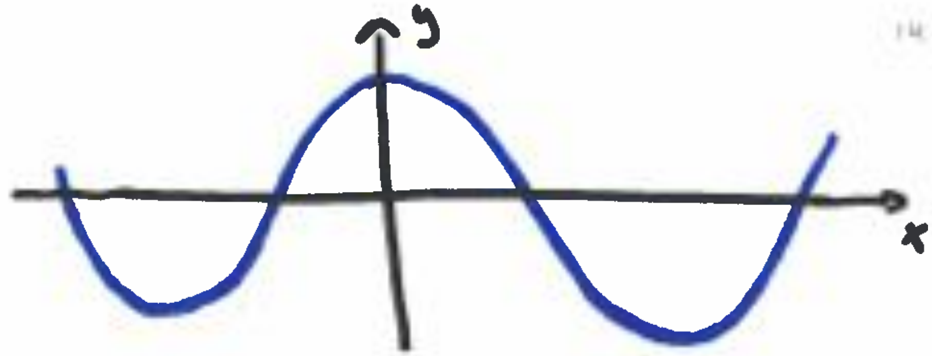
⚠ For the remainder of §2.3, a broken document camera forced the lecture onto the white board. I'm still posting this "outline" as a reminder of what we covered.

Functions Acting on Sets

Def Let $f: A \to B$ and suppose $C \subseteq A$, $D \subseteq B$. Then



__Ex__ $f(x) = \cos x$



$f^{-1}(\{1\}) =$

$f^{-1}((0,1])$

$f([\pi/2, 4\pi/3))$

$\exists$ *many* theorems involving all of these concepts:

Thm 2.3.16

(a) $C \subseteq f^{-1}[f(C)]$

(b) $f[f^{-1}(D)] \subseteq D$

(c) $f(C_1 \cap C_2) \subseteq f(C_1) \cap f(C_2)$

$\vdots$

Thm 2.3.18

$\vdots$

(c) If f is injective, $f(C_1 \cap C_2) = f(C_1) \cap f(C_2)$

Last Concept in §2.3

Def Let $f: A \to B$ be bijective. The **inverse of f** is

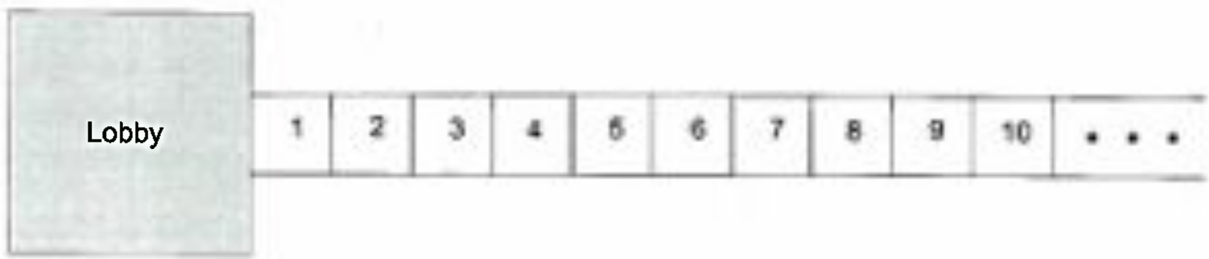
Thm 2.3.24 Let $f: A \to B$ be bijective. Then

(a) $f^{-1}: B \to A$

(b) $f^{-1} \circ f = id_A$ and

Hotel Infinity

After years of working your way up through the ranks, you've finally achieved your goal: you're the new manager of the Hotel Infinity! You are especially excited because the Hotel Infinity is one of a kind: it has infinitely many rooms, one for each whole number, arranged in an infinitely long hallway stretching off from the lobby:



On your first day, the owner asks if you have any questions.

"Just one," you say. "When I was looking around, I noticed that there's no way to change the VACANCY sign to NO VACANCY."

"Ah yes," says the owner, "I'm glad you noticed. That's by design; if you run this hotel correctly, you'll never need a NO VACANCY sign. If you ever turn a lodger away, I promise you: it will be your last day working at this hotel!"