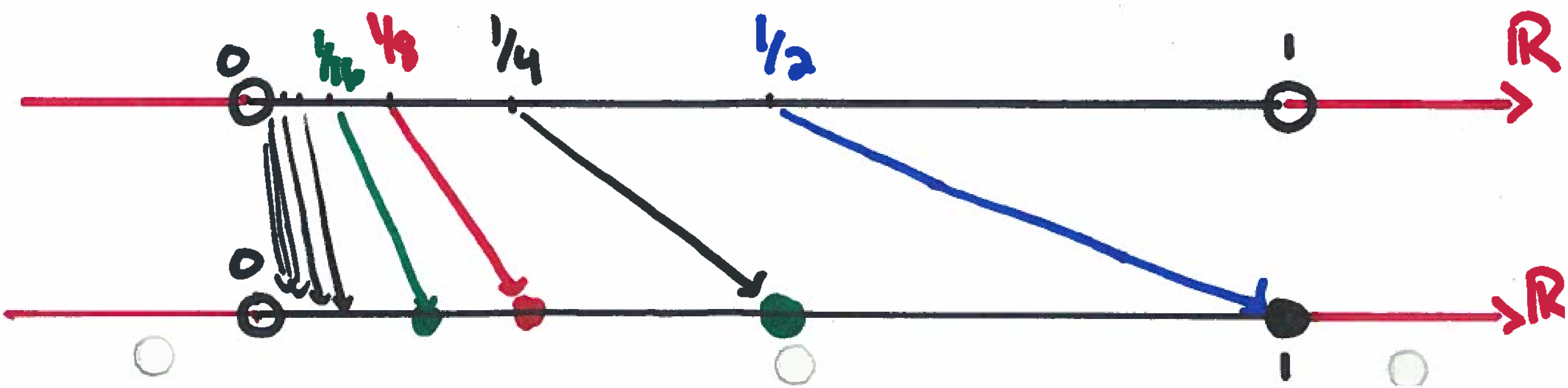
Ex $(0,1) \sim (0,1]$

idea mostly use $\mathrm{id}: (0,1) \to (0,1]$ which sends x to x

but need to send something to 1 (use $1/2$)

creates a hole at $1/2$, so send $1/4 \mapsto 1/2$, repeat.

Define $f: (0,1) \to (0,1]$, $f(x) = \begin{cases} x, & x \neq \frac{1}{2^n}, \text{ some } n, \\ \underset{\substack{\| \\ 2x}}{\frac{1}{2^{n-1}}}, & x = \frac{1}{2^n}, \ n \in \mathbb{N} \end{cases}$



Thm 2.4.10 TFAE ("The following are equivalent")

(a) S is ctble
(b) $\exists$ injection $f: S \to \mathbb{N}$.
(c) $\exists$ surjection $g: \mathbb{N} \to S$.

i.e. $a \iff b$
$a \iff c$
$b \iff c$

3 iff stmts, so 6 pfs.

We commonly prove
$$\begin{array}{ccc} (a) & \Rightarrow & (b) \\ \Uparrow & & \Downarrow \\ & (c) & \end{array}$$
(3 pfs)

or $(a) \iff (b) \iff (c)$ (4 pfs)

Sketch of (parts) of pf.

$(a) \Rightarrow (c)$. S ct'ble $\Rightarrow S = \{s_1, \ldots, s_n\}$ or $S = \{s_1, s_2, \ldots\}$

$\underline{S \text{ finite}}$ define $g: \mathbb{N} \to S$, $g(1) = s_1, \ldots, g(n) = s_n$,

$g(m) = s_1$, $m > n$

$\underline{S \text{ infinite}}$ $g(n) = s_n$

Putting it all together

If $S \sim T$, they have the same cardinality.

So $\{1,2,3\}$ and $\{a,b,c\}$ not equal as sets, but same cardinality.

The cardinal number or cardinality of a set is (informally) its size.

- cardinal # of $\emptyset$ is $|\emptyset| = 0$
- —— " —— $I_n = \{1,2,3,\dots,n\}$ is $|I_n| = n$.
- $|\mathbb{N}| = \aleph_0$
- $|\mathbb{R}| = c$ ← continuum

Def 2.4.14 Denote card'l number of S by $|S|$, so $|S|=|T|$ iff $S \sim T$, i.e. $\exists$ bijection $f: S \to T$.

- Define $|S| \leq |T|$ if $\exists$ injection $S \to T$.
- Define $|S| < |T|$ if $|S| \leq |T|$ but not $|S| = |T|$.
  - └ need to define b/c it's not $<$ with $\mathbb{R}$; $|S|, |T|$ could be ∞

Working with these def$^{\underline{n}}$'s not always as awful as it seems at first.

Thm 2.4.15 Let S, T, U be sets.

(a) $S \subseteq T \Rightarrow |S| \le |T|$

$$\left.\begin{array}{l} f: S \longrightarrow T \\ s \longmapsto s \end{array}\right\}$$ this "identity" fn always an injection

$\Rightarrow$ by def^n, $|S| \le |T|$.

(b) $|S| \le |S|$

$S \subseteq S \Rightarrow |S| \le |S|$ by (a).

(c) $\underbrace{|S| \le |T|}_{\exists \text{ inj } f: S \to T}$ and $\underbrace{|T| \le |U|}_{\exists \text{ inj } g: T \to U}$, then $|S| \le |U|$

$\Rightarrow g \circ f : S \longrightarrow U$ is inj'n

$\Rightarrow |S| \le |U|$

(d) $m, n \in \mathbb{N}$, $m \le n$, then $|I_m| \le |I_n|$.

$m \le n \Rightarrow \{1, 2, 3, \ldots, m\} \subseteq \{1, 2, 3, \ldots, m, m+1, \ldots, n\}$

use (a) again.

(e) [You read]

∞ many ∞'s....

Def Given a set S, the power set of S, written $\mathcal{P}(S)$, is the set of all subsets of S.

Ex $S = \{a,b,c,d\}$

	0 elt subset	1-elt subsets	2-elt subsets	3 elt	4 elt subset	
$\mathcal{P}(S) = \{$	ϕ,	$\{a\}$,	$\{a,b\}$,	$\{b,c,d\}$,	$\{a,b,c,d\}$	$\}$
		$\{b\}$,	$\{a,c\}$,	$\{a,c,d\}$,		
		$\{c\}$,	$\{a,d\}$,	$\{a,b,d\}$,		
		$\{d\}$,	$\{b,c\}$,	$\{a,b,c\}$,		
			$\{b,d\}$,			
			$\{c,d\}$,			

Another way to count (not list them)

To construct a subset of $\{a, b, c, d\}$:

- For each elt, I need to decide whether it's included or not - Four Y/N questions

Total possibilities: $2^4 = 16$ ← # of elts in orig set

Y/N is two options

<u>Thms.</u>

- $|S| < |\mathcal{P}(S)|$
- $|\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$ (i.e. $2^{\aleph_0} = c$)

<u>Corollary</u> $\exists$ infinite "chain" of larger and larger ∞'s:

$$|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < \cdots$$

One last curiosity

If $|S| \le |T|$ and $|T| \le |S|$, can we conclude that $|S| = |T|$?

Yes, but not by def$^{\underline{n}}$ - by Schröder-Bernstein Thm

Ex $(0,1) \sim (0,1]$

$$\begin{aligned} id_{(0,1)} : (0,1) &\longrightarrow (0,1] \\ x &\longmapsto x \end{aligned}$$ is inj'n $\Rightarrow |(0,1)| \le |(0,1]|$

$$\begin{aligned} f : (0,1] &\longrightarrow (0,1) \\ x &\longmapsto \tfrac{x}{2} \end{aligned}$$ not surj, is inj

$\Rightarrow |(0,1]| \le |(0,1)|$

$\Rightarrow |(0,1)| = |(0,1]|$ by S.B. Thm.

A few words about §2.5, which covers Axioms of Set Theory

Math majors encouraged to read this section, but it's not officially part of this course.

Has the basic Axioms we use to build up set thy and modern mathematics.

Paradoxes: Some sets are members of other sets ($S \in \mathcal{P}(S)$), even of themselves. Define

$$B = \{S : S \notin S\}$$

Is B an elt of itself? (i.e. is $B \in B$)