# §2.4 Cardinality

Ok, the Hotel Infinity teaches us that infinity is weird. Especially when we compare sizes of infinite sets.

For finite sets it's easier! If $A=\{1,2\}$ and $B=\{0,\triangle,\square\}$ then A has 2 elts and B has 3, so B is "larger."

**Ex** Write these sets in order from "smallest" (i.e. fewest members) to "largest" (most elts).

$\mathbb{N} = \{1, 2, 3, \dots\}$

$\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$

$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$

$\mathbb{Q}^+ = \{\frac{p}{q} \geq 0 \text{ etc}\}$

$\mathbb{Q}$

all turn out to be same "size"

Irrationals $= \mathbb{R} \setminus \mathbb{Q}$

$\mathbb{R}$

$\mathbb{C}$

$(0,1)$

$[0,1)$

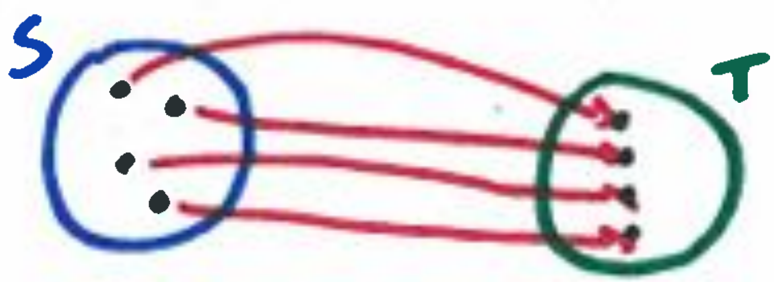
$[0,1]$

all turn out to be same "size"

But the sets in left col. have a different "size" than those on the right.

## BIJECTIONS bring order to this chaos.

**Def** Two sets $S, T$ are equinumerous if $\exists$ bijection $S \to T$. Write: $S \sim T$




everything in T is "hit" by exactly one elt.

idea if $S \sim T$, they are in 1:1 correspondance and are the "same set" (with elts relabeled), hence the same size.

<u>Ex</u> $\{1,2,3\}$ and $\{a,b,c\}$

Define $f(1) = a$, $f(2) = b$, $f(3) = c$. $f$ is bij'n by inspection, so $\{1,2,3\} \sim \{a,b,c\}$

$\{1,2,3,4\}$ and $\{a,b,c\}$

<u>No bij'n possible</u>, so these sets are different sizes.

↳ could choose distinct outputs for $f(1), f(2), f(3)$, but $f(4)$ will break injectivity.

$\{1,2,3,4\}$ and $\mathbb{N}$

<u>No bij'n possible</u>

↳ we can choose up to 4 different outputs, but will never be surjective.

**Ex** $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$

$\mathbb{N} = \{1, 2, 3, \dots\}$

$f: \mathbb{N}_0 \to \mathbb{N}$, $f(n) = n+1$.

Is $f$ surj? Let $m \in \mathbb{N}$. Then

$m = f(m-1) = (m-1)+1$.

m, n would be better than x, y... $\longrightarrow$ Is $f$ inj? Let $x, y \in \mathbb{N}_0$, $x \neq y$.

Then $x-1 \neq y-1$

$\Rightarrow f(x) \neq f(y)$

**Ex** $\mathbb{N}, \mathbb{Z}$

**Want** $n, f(n)$ for $f: \mathbb{N} \to \mathbb{Z}$.

| $n$ | $f(n)$ |
|---|---|
| 1 | 0 |
| 2 | 1 |
| 3 | -1 |
| 4 | 2 |
| 5 | -2 |
| 6 | 3 |
| 7 | -3 |
| ⋮ | ⋮ |

$f(n) = (-1)^n \left\lfloor \frac{n}{2} \right\rfloor$ – floor (round down)

Ex $\mathbb{N}$, $\mathbb{Q}^+ = \{\frac{p}{q} \geq 0\}$

start

$$\begin{array}{cccccc}
\frac{0}{1}, & \frac{0}{2}, & \frac{0}{3}, & \frac{0}{4}, & \frac{0}{5} & \cdots \\
\frac{1}{1}, & \frac{1}{2}, & \frac{1}{3}, & \frac{1}{4}, & \frac{1}{5}, & \cdots \\
\frac{2}{1}, & \frac{2}{2}, & \frac{2}{3}, & \frac{2}{4}, & \frac{2}{5}, & \cdots \\
\frac{3}{1}, & \frac{3}{2}, & \frac{3}{3}, & \cdots & & \\
\vdots & \vdots & \vdots & & &
\end{array}$$

$\frac{p}{q}$ is in $(p+1)^{\text{st}}$ row ($p=0$ possible)

$q^{\text{th}}$-col

**Define** $f: \mathbb{N} \to \mathbb{Q}^+$ as follows:

$f(n) = n^{\text{th}}$ unique # we meet on this path.

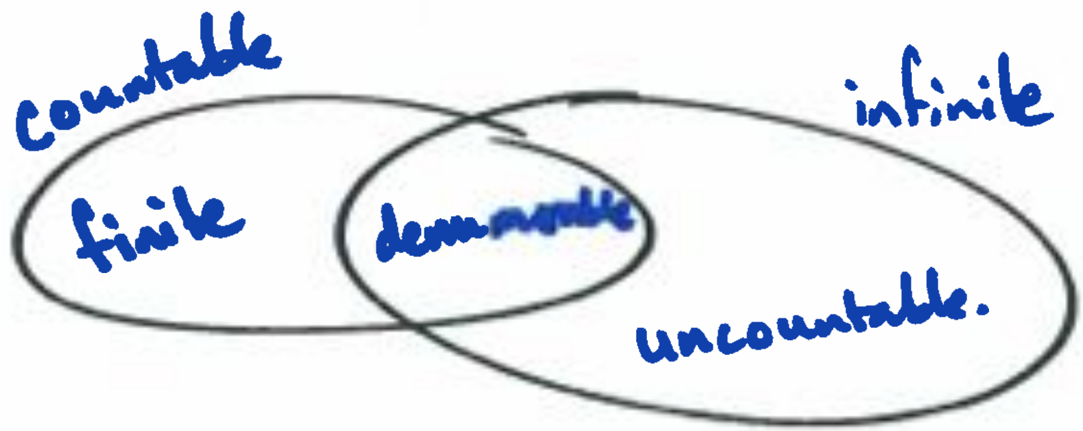
(We wrote $f(1), f(2), \ldots, f(7)$ on board and discussed why this is a bijection.)

**Def** A set $S$ is ....

- **finite** if $S \sim I_n = \{1, 2, \ldots, n\}$
- **denumerable** if $S \sim \mathbb{N}$
- **countable** if $S$ is finite or denumerable.
- **uncountable** if $S$ is not countable.

$I_n$

$\exists$ handy-dandy Venn Diagram in your book (p84)



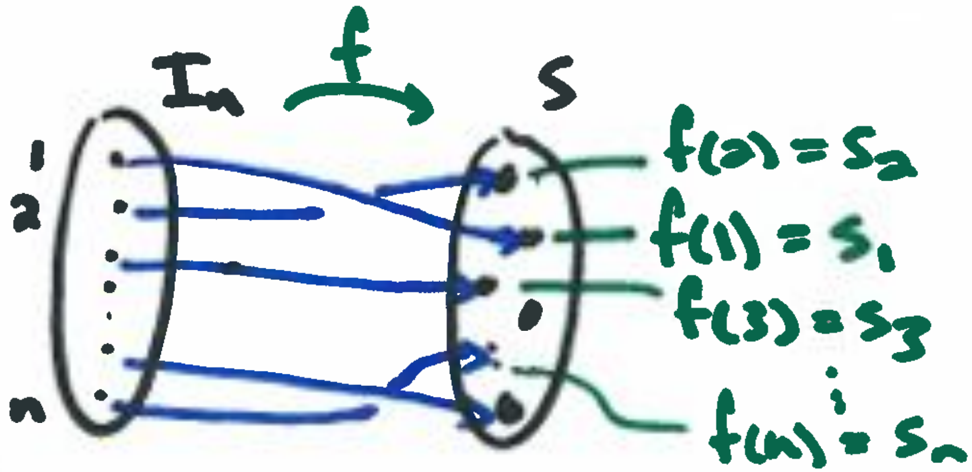
# Useful Consequence

Countable sets can be "listed in order"

**S finite**: can write $S = \{s_1, s_2, s_3, \dots, s_n\}$

**S denumerable**: $S = \{s_1, s_2, s_3, s_4, s_5, \dots\}$

**Ex** $I_n \sim S$



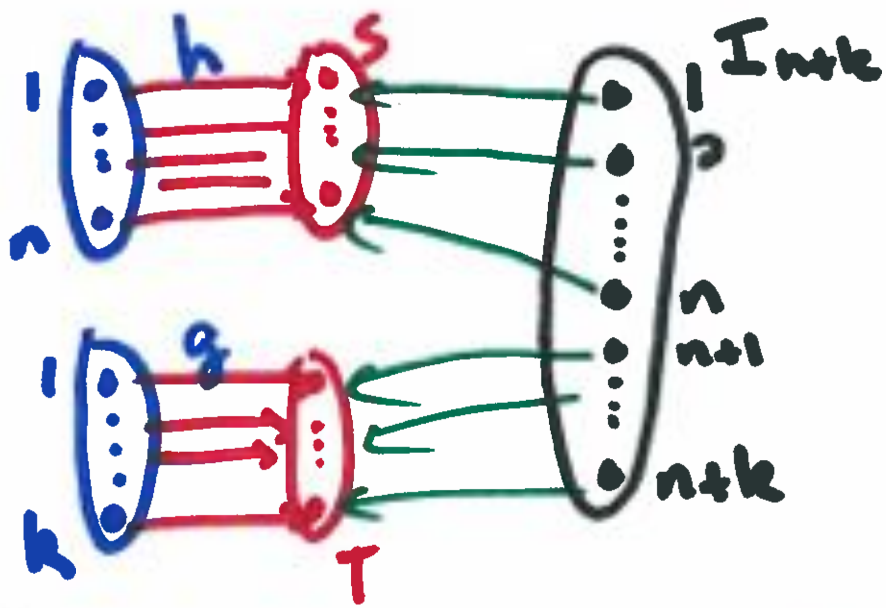

⚠ No "canonical" order for which elt is $s_1$, which is $s_2$, etc.

**Thm** (Ex 2.4.11) Let $S, T$ be countable. Then $S \cup T$ is countable.

**Pf** $\exists$ 3 cases:

1. $S, T$ both finite.
2. One is finite, one is denumerable
3. Both are denumerable.

**Case 1** $S \sim I_n$, $T \sim I_k$




**Define** $f : I_{n+k} \to S \cup T$

$$f(p) = \begin{cases} h(p) & , p \le n \\ g(p-n) & , p > n \end{cases}$$

**Case 2** WLOG (without loss of generality) assume $S$ is finite, $T$ denumerable.

$$S \cup T = \underbrace{\{s_1, s_2, \dots, s_n, t_1, t_2, t_3, t_4, \dots\}}_{\text{equinumerous with } \mathbb{N}.}$$

**Case 3** Suppose $f:\mathbb{N}\to S$, $g:\mathbb{N}\to T$ are bij'n.

define $h:\mathbb{N}\to S\cup T$ by

$$h(n) = \begin{cases} f((n+1)/2), & n \text{ odd} \\ g(n/2), & n \text{ even} \end{cases}$$

$\triangle$! Left for you in each case: what if $S\cap T \neq \emptyset$?

**Thm** (Practice 2.4.2) $\sim$ is an equiv. reln.

**Refl** $id_X : X \longrightarrow X$ bij'n. **Sym**: $f: X \to Y$ bij'n, $f^{-1}: Y \to X$ bij'n. **Trans**: composition of bij'ns is bij

$x \longmapsto x$

**Corollary** $\mathbb{N} \sim \mathbb{Q}$

**Pf**: $\mathbb{N} \sim \mathbb{Q}^+ \sim (\mathbb{Q}^+ \cup \{0\}) \sim \underbrace{(\mathbb{Q}^+ \cup \{0\}) \cup \mathbb{Q}^-}_{= \mathbb{Q}}$

($\mathbb{N} \sim \mathbb{Q}^-$ by same "path" argument as $\mathbb{Q}^+$...)

**Thm 2.4.3** Any subset of a countable set $S$ is countable

Thm 2.4.12 $\mathbb{R}$ is uncountable

(Part of your intellectual heritage!)

Pf: Using CP of previous thm, we'll show $(0,1)$ is uncountable $\Rightarrow$ $\mathbb{R}$ is uncountable.

Assume $(0,1)$ is countable, so we can list its elts. in order:

$$x_1 = 0.\,x_{11}\,x_{12}\,x_{13}\,x_{14}\,x_{15}\,x_{16}\cdots\cdots$$
$$x_2 = 0.\,x_{21}\,x_{22}\,x_{23}\,x_{24}\,x_{25}\cdots$$
$$x_3 = 0.\,x_{31}\,x_{32}\,x_{33}\,x_{34}\cdots$$
$$x_4 = 0.\,x_{41}\,x_{42}\,x_{43}\,x_{44}\cdots$$
$$\vdots$$

Define $b = 0.b_1 b_2 b_3 b_4 b_5 \cdots$ by $b_n = \begin{cases} 2, & x_{nn} \neq 2 \\ 3, & x_{nn} = 2 \end{cases}$

$\Rightarrow b$ is not in my list by construction, even though $b \in (0,1)$ $\lightning$.