Chapter 3: The Real Numbers

§3.1 $\mathbb{N}$ (?!) and induction

$\mathbb{N}$ provides a nice intro to properties of number systems and sets. For example:

Axiom 3.1.1 $\mathbb{N}$ is well ordered, which means if $\emptyset \neq S \subseteq \mathbb{N}$ then S has a "least" element, i.e. $\exists\, k \in S$ such that $k \leq m \;\forall\, m \in S$.

Ex $S = \{10, 9, 100, 99, 1000, 999, \ldots\} \subseteq \mathbb{N}$.

$9 \leq m \ \forall m \in S$.

Aside #1 Could look at the rest of the elts in S (assuming there are any left over). Choose the least elt of those remaining. Repeat, eventually put the set in order:

$S = \{9, 10, 99, 100, 999, 1000, \ldots\}$

Aside #2 Can every set be well ordered?

What's the minimal elt in $(0,1)$?

Well Ordering "Theorem" Every set can be well-ordered (with respect to <u>some</u> order - not necessarily $<$, $\leq$, etc.)

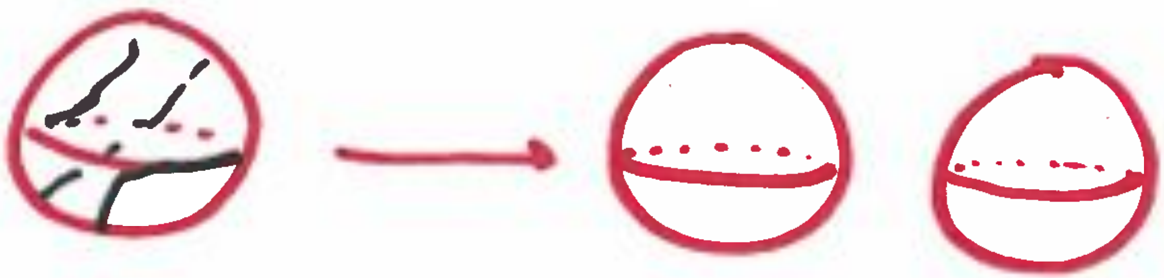
Hmm... believable? Equivalent to...

Axiom of Choice Given any infinite collection of bins (sets) we can choose one object (elts) from each.

Seems more reasonable (?). Has lots of useful, reasonable consequences, but <u>also</u> some odd ones, like WOT above and....

Banach Tarski Paradox

A sphere in $\mathbb{R}^3$ can be cut into a finite number of pieces, which can be rearranged and glued back together.... into two identical copies of the original sphere. (!!?!)



Not physically possible - the sets require so'ly jagged cuts, smaller than size of atoms...


I CARVED AND CARVED, AND THE NEXT THING I KNEW I HAD TWO PUMPKINS.
I TOLD YOU NOT TO TAKE THE AXIOM OF CHOICE.

Back to "Reality"...

Thm 3.1.2 (Proof by Induction)

Let $P(n)$ be a stmt which is T/F for each n.

If (a) $P(1)$ is true. [base / anchor]

(b) $\forall k \in \mathbb{N}$, if $P(k)$ true $\Rightarrow P(k+1)$ true [induction step]

Then $P(n)$ true for all $n \in \mathbb{N}$.

Pf: Let $S = \{k \mid P(k) \text{ is false}\}$. If $S = \emptyset$ then $P(n)$ true $\forall n$, done. If $S \neq \emptyset$, then it has a least element m. (by well ordering)

By (a) [base case], $m \neq 1$, so $m > 1$ and $m-1 \in \mathbb{N}$. $P(m-1)$ true, so (b) [ind step] $\Rightarrow P(m)$ true. $\lightning$.

Obligatory Historical Example

Prove: $1+2+3+4+\cdots+n = \frac{n(n+1)}{2}$: $P(n)$

Check $P(1)$: $1 = \frac{1(2)}{2} = \frac{2}{2} = 1$ ✓

$P(5)$: $1+2+3+4+5 = 15$

$\frac{5(6)}{2} = \frac{30}{2} = 15$ ✓

$$
\begin{array}{l}
P(1000):\ 1+2+3+\cdots+998+999+1000 \\
\quad\ 1000+999+998+\cdots+3+2+1 \\
\hline
\quad\ 1001+1001+\cdots+1001
\end{array}
$$

Answer: $\frac{1000\cdot(1001)}{2}$

Inductive Proof

$P(n): 1+2+3+\cdots+n = \frac{n(n+1)}{2}$

Base Case: $P(1)$ verified above.

Induction Step: Assume $P(k)$ is true,

$$1+2+3+\cdots+k = \frac{k(k+1)}{2}.$$

Show $P(k+1)$ is true.

$$(1+2+3+\cdots+k-1+k)+k+1 = \underline{(1+\cdots+k)}+(k+1)$$

$$= \frac{k(k+1)}{2}+(k+1)$$

$$= \frac{k^2+k+2k+2}{2}$$

$$= \frac{k^2+3k+2}{2}$$

$$= (k+1)(k+2)/2$$

⚠ What's wrong with this "inductive proof"?

Base Case $n=1$: $1 = \frac{1(2)}{2}$ ✓

Induction Step: Assume true for n, prove for $n+1$:

$$1+2+3+\cdots+(n-1)+n+(n+1) = \frac{(n+1)(n+2)}{2}$$

$$1+2+3+\cdots+n = \frac{(n+1)(n+2)}{2} - (n+1)$$

$$1+2+3+\cdots+n = \frac{(n^2+3n+2)-2n-2}{2}$$

$$1+2+3+\cdots+n = \frac{n^2+n}{2}$$

$$1+2+3+\cdots+n = \frac{n(n+1)}{2}$$, which is true by assumption.