§3.2 Ordered Fields

You should <u>read</u> this section, but you are not responsible for all of it.

Focus on the def$\underline{\text{s}}$ discussed in lecture, the few thms we'll mention, and any HW assigned.

The rest is "ambience", showing just how much is involved in constructing algebra...

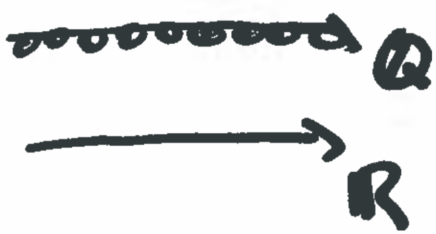
$\mathbb{R}$ is a "complete ordered field," meaning:

Complete - will be described in next section.

Essentially means $\mathbb{R}$ has no "holes."

Ex1 in $\mathbb{Z}$, no #s b/w 0, 1.

in $\mathbb{R}$, $[0,1]$ is uncountable.



Ex2 In $\mathbb{Q}$, there is no # s.t. $q^2 = 2$.

But $\sqrt{2} \in \mathbb{R}$.

ordered Given any two real #'s x ~~any~~ and y, we can place them in order: $x \leq y$, $x > y$, ...

field means we can do all standard arithmetic, $+, -, \cdot, \div$ (but not by 0).

Examples

$\mathbb{N}$: $\forall\ m, n \in \mathbb{N}$, $m+n \in \mathbb{N}$. (we can add)

$10 - 15 = -5 \in \mathbb{Z}$, <u>not</u> <u>in</u> $\mathbb{N}$. (can't do $-$).

Can multiply, but not divide in gen'l.

$\mathbb{Z}$: we can add, subtract, multiply two integers and get an integer. <u>BUT</u>....

in gen'l, can't divide: $\frac{1}{2} \notin \mathbb{Z}$.

$\mathbb{Q}$: add, subtract, multiply divide (non zero #) fractions and get fractions: <u>field</u>.

Thm 3.2.2 Let $x, y, z \in \mathbb{R}$

(a) $x + z = y + z \Rightarrow x = y$

(b) $x \cdot 0 = 0$

(c) $-0 = 0$

(d) $(-1) \cdot x = (-x)$

(e) $xy = 0 \iff x = 0$ or $y = 0$.

(f) $x < y \iff -y < -x$

(g) $x < y$ and $z < 0 \Rightarrow zy < zx$.

(A4) $\exists!\ 0 \in \mathbb{R} \ni \forall x \in \mathbb{R},\ x + 0 = x$.

(DL) $x(y+z) = xy + xz$

(A2) $x + y = y + x$

Pf of (b)

$$x \cdot 0 = x \cdot (0 + 0) \quad \text{(A4)}$$

$$x \cdot 0 = x \cdot 0 + x \cdot 0 \quad \text{(DL)}$$

$$x \cdot 0 + 0 = x \cdot 0 + x \cdot 0 \quad \text{(A4)}$$

$$0 + \cancel{x \cdot 0} = x \cdot 0 + \cancel{x \cdot 0} \quad \text{(A2)}$$

$$0 = x \cdot 0 \quad \text{Part (a)}$$

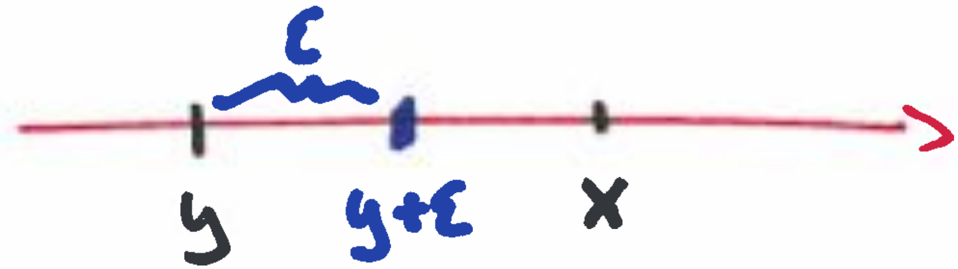
Thm 3.2.8 If $x \le y+\varepsilon \ \forall \varepsilon>0$, then $x \le y$.

Remark Think "limits" — $y+\varepsilon \ \forall \varepsilon>0$ is fancy way of writing $\lim_{h\to 0^+} y+h$

Thm "says" $\lim_{h\to 0^+} x \le \lim_{h\to 0^+} y+h \Rightarrow x \le y$.

CP: if $y<x$ then $\exists\, \varepsilon>0$ such that $x>y+\varepsilon$.

Pf of: if $y<x$ then $\exists \varepsilon>0 \ni x > y+\varepsilon$



From picture, we could use

$$y+\varepsilon = \frac{x+y}{2} \Rightarrow \varepsilon = \frac{x-y}{2}.$$

Check:
$$\begin{aligned} y+\varepsilon &= y + \frac{x-y}{2} \\ &= \frac{2y+x-y}{2} = \frac{x+y}{2} \\ &< \frac{x+x}{2} \quad (\text{b/c } x>y) \\ &= \frac{2x}{2} = x. \end{aligned}$$

Thm 3.2.10

(a) $|x| \geq 0$

(b) $|x| \leq a \iff -a \leq x \leq a$

(c) $|xy| = |x| \cdot |y|$

(d) $|x+y| \leq |x| + |y|$

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Pf (a) Case 1: $x \geq 0$, so $|x| = x \geq 0$ ✓

Case 2: $x < 0$, so $|x| = -x = (-1) \cdot x > 0$ using order axioms + thms. ✓

(d) We can say $-|x| \leq x \leq |x|$

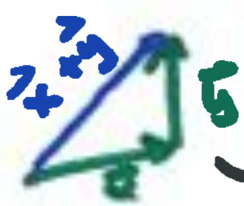
$-|y| \leq y \leq |y|$

add together: $-|x| - |y| \leq x+y \leq |x| + |y|$

$-(|x| + |y|) \leq x+y \leq (|x| + |y|)$

$|x+y| \leq |x| + |y|$ (b).

$\triangle$ ineq: name clearer w/ vectors:



$\sim |\vec{x}+\vec{y}| \leq |\vec{x}| + |\vec{y}|$