§3.3 The Completeness Axiom

Another section with a lot of topics:

- supremum, infimum, bounds
- Completeness Axiom
- Archimedean Property of $\mathbb{R}$
- Density

and.... "style" of proofs w/ sup, inf. (w/ ε...)

Bounds

Ex What are the max, min elts of....

$S = \{0, 2, 4\}$ min - 0, max - 4

$S = [0, \infty)$ min - 0, no max.

$S = (0, 1)$ no min, no max

$S = \{1 - \frac{1}{n} \mid n \in \mathbb{N}\} = \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\}$ min - 0, no max

Def $m \in S$ is min S if $m \leq s \ \forall s \in S$.

$M \in S$ is max S if $s \leq M \ \forall s \in S$.

More generally....

m lower bound for S if $m \leq s \ \forall s \in S$

M upper bound for S if $s \leq M \ \forall s \in S$.

Ex

$S = \{0, 2, 4\}$ 0 is min and lower bd. $-1, -10, -\pi,$ -10^{26} also lower bds.
4 is max, hence U.B.
other bds include $5, 10, 20, \ldots$

$S = [0, 1)$ no max, but 1 is upper bound (so is $2, 10, 39 \ldots$)

$S = \{1 - 1/n \mid n \in \mathbb{N}\}$

Observations

1. m is an upper bound for S. $\Rightarrow$ so is any # larger than m.

2. Similar for Lower bds: if m is L.B. for S, so is any # $< m$.

3. min/max automatically a LB/UB

   Conversely, a LB/UB in the set is automatically a min/max.

Q: can we find the "best" LB, UB for S?

Def Let $\emptyset \neq S \subseteq \mathbb{R}$. If S is bounded above, then the <u>supremum</u> of S is its <u>least upper bound</u>, denoted: $\sup S = \text{lub}\, S$.

Hence $m = \sup S$ iff:

(a) m is upper bound: $\boxed{m \geq s \ \forall s \in S}$

(b) anything smaller than m is <u>not</u> an upper bd:

If $\forall m' < m$ it is not an upper bd, i.e. if

$$\boxed{m' < m \text{ then } \exists\, s' \in S \text{ s.t. } m' < s'}$$

Ex $S = (0,1)$. Claim that $1 = \sup S$.

(a) $S = \{0 < x < 1\} \Rightarrow 1$ is upper bound.

(b) any # less than 1 is not least u.b.

If $u \in (0,1)$, then $\frac{u+1}{2} \in (0,1)$ is bigger than u.

not rigorous yet.

Similarly, if $S \subseteq \mathbb{R}$ is non empty and bdd below, its greatest lower bound or infumum ($\inf S = glb S$)

Satisfies
(a) its a lower bound
(b) no larger # is a lower bound.

Ex $S = [0,1]$

$\inf S = 0$
$\sup S = 1$

$S = (0, \infty)$

$\inf S = 0$
$\sup S$ does not exist.
(S has no upper bound, hence no least upper bound.)

$S = \{\frac{1}{n} \mid n \in \mathbb{N}\} = \{1, 1/2, 1/3, 1/4, \ldots\}$

$\inf S = 0$
$\sup S = 1.$

☆ You can often find $\inf S$, $\sup S$ intuitively or "by inspection." Proving your answers are correct is trickier, especially part (b) of the def$^{\underline{n}}$.

Ex to prove $m = \sup S$, must 1$^{\underline{st}}$ show m is an upper bd of S – usually with algebra.

Then must show any smaller m' (i.e. $m' < m$) is not an upper bd.

$\boxed{☆☆☆} \to$ OFTEN we choose small $\varepsilon > 0$, set $m' = m - \varepsilon$

Ex (Spring 2010 Exam Prob) $A = \{\frac{n-1}{n+1} \mid n \in \mathbb{N}\}$

So $A = \{\frac{0}{2}, \frac{1}{3}, \frac{2}{4}, \frac{3}{5}, \frac{4}{6}, \frac{5}{7}, \ldots\}$ $\left(= \{0, 1/3, 1/2, 3/5, 2/3, \ldots\}\right)$

(i) Show A bdd below, above.

$n \in \mathbb{N} \Rightarrow n-1, n+1$ both $\geq 0 \Rightarrow \frac{n-1}{n+1} \geq 0$. Hence A bdd below by 0.

Also, $\forall n$, $n-1 < n+1 \Rightarrow \frac{n-1}{n+1} < 1$. Hence A bdd above by 1.

(ii) Find $\ell = \inf A$, $m = \sup A$. (Prove your answers!)

$\ell = \inf A = 0$, $m = \sup A = 1$.

Proof that $\inf A = 0$. First, as shown in (i), $\ell = 0$ is a lower bound of A, because $0 \leq a$ $\forall a \in A$. But it is also the greatest lower bound, because any larger number will not be a lower bound: If $\ell' > 0$, then it's not a lower bd because 0 itself is in A and less than ℓ'!

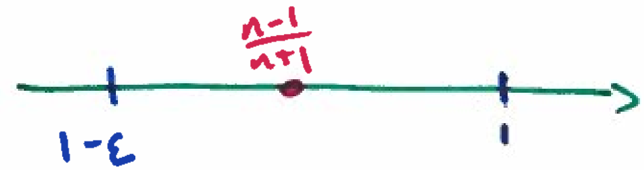
<u>Proof that $m = \sup A = 1$:</u> From (i) we know $m=1$ is upper bound.

To prove it's sup A, we must show any # <u>less</u> <u>than</u> $m=1$ cannot be an upper bound.

Take (presumably small) $\varepsilon > 0$, so $1-\varepsilon < 1$, and show

$\exists$ $n \in \mathbb{N}$ such that $1-\varepsilon < \underbrace{\frac{n-1}{n+1}}_{\text{elt of } A} < 1$.

elt of A, so $1-\varepsilon$ not an upper bound.



If that's true for any $\varepsilon > 0$, then we have shown no # less than 1 can be upper bound, hence $m=1$ <u>is</u> sup A.

These types of proofs usually have a "Think" portion and a "proof" portion.

"scratchwork"
"Algebra"
etc.

Repeat to show $\sup A = 1$, show: $\forall \varepsilon > 0\ \exists n \in \mathbb{N}$ s.t. $1 - \varepsilon < \frac{n-1}{n+1} (\leq 1)$
which implies $1-\varepsilon$ not upper bound.

automatic from (i)

Use algebra to "solve" for n in terms of ε:

$$1 - \varepsilon < \frac{n-1}{n+1} = \frac{(n-1+2)-2}{n+1} = \frac{(n+1)-2}{(n+1)} = 1 - \frac{2}{n+1}$$

"Think" Portion of this Problem

$$-\varepsilon < -\frac{2}{n+1}$$

$$\varepsilon > \frac{2}{n+1}$$

$$n+1 > \frac{2}{\varepsilon}$$ ($n+1$, ε both >0 so ineq doesn't change direction...)

$$n > \frac{2}{\varepsilon} - 1$$

So any $\frac{n-1}{n+1}$ (with $n > \frac{2}{\varepsilon} - 1$) will be larger than $1-\varepsilon$,
so $1-\varepsilon$ not an upper bound.

⚠ For actual proof, use what we found in algebra and write it in reverse!

Proof that $\sup\underbrace{\left\{\frac{n-1}{n+1} : n \in \mathbb{N}\right\}}_{=A} = 1$:

Pf First note that, $\forall\, n \in \mathbb{N}$, $n-1 < n+1$. Hence $\frac{n-1}{n+1} < 1 \Rightarrow 1$ is an upper bound of A.

"Proof" Portion

To show 1 is the supremum of A, we must also show that $\forall \varepsilon > 0$, $1-\varepsilon$ is not an upper bound of A.

Given any $\varepsilon > 0$, choose $n > \frac{2}{\varepsilon} - 1$, which is possible by the Arch. Prop.

Then
$$n+1 > 2/\varepsilon$$
$$\varepsilon > \frac{2}{n+1}$$
$$-\varepsilon < -\frac{2}{n+1}$$
$$1-\varepsilon < 1 - \frac{2}{n+1} = \cdots = \frac{n-1}{n+1} \in A.$$
$\Rightarrow 1-\varepsilon$ is not an upper bound of A. ∎

$\mathbb{R}$ is "complete", which means it satisfies:

Completeness Axiom Every non-empty subset of $\mathbb{R}$ which is bounded above has a supremum.

Notes ① $\mathbb{Q}$ not complete.

Ex $S = \{q \in \mathbb{Q} \mid q^2 < 2\} \subseteq \mathbb{Q}$ is non-empty (Check: $0 \in S$) and bounded above (Check: 2 is an upper bound. So is 10, or 3.141...) but $\sup S$ would be $\sqrt{2} \notin \mathbb{Q}$.

② Also implies every non-empty subset of $\mathbb{R}$ which is bounded below has an infimum. (Can you see why? Hint: def $T = \{-s \mid s \in S\}$ to convert "bdd below" to "bdd above.")

<u>Thm 3.3.9</u> (Archimedean Property of $\mathbb{R}$)

$\mathbb{N}$ is unbounded in $\mathbb{R}$.

(Pf in book)

Thm 3.3.10 TFAE

(*) Archimedean Property

* (a) $\forall z \in \mathbb{R}\ \exists n \in \mathbb{N}$ s.t. $n > z$

(b) $\forall x > 0\ \forall y \in \mathbb{R},\ \exists n$ s.t. $nx > y$.

* (c) $\forall x > 0\ \exists n \in \mathbb{N}$ s.t. $0 < \frac{1}{n} < x$.

To save time, prove (*) $\Rightarrow$ (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (*)

(*) $\Rightarrow$ (a) prove by contradiction. If no such n exists, then z would be an upper bd of $\mathbb{N}$. $\lightning$

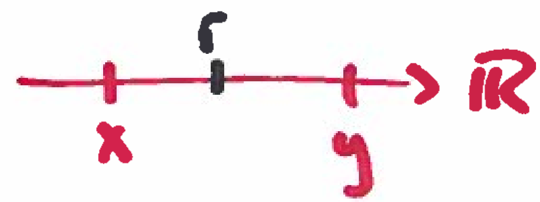
(a) $\Rightarrow$ (b) Choose $z = y/x$ and apply (a). $n > z$, $n > \frac{y}{x}$, $nx > y$

(b) $\Rightarrow$ (c) Let $y = 1$ in (b): $nx > 1$, so $x > \frac{1}{n}$

(c) $\Rightarrow$ (*) (You think about)

Finally ....

Thm 3.3.13 $\mathbb{Q}$ is dense in $\mathbb{R}$: $\forall\, x, y \in \mathbb{R}$, $x<y$, there exist $r \in \mathbb{Q} \ni x<r<y$:



Pf: Read Book.

Constructive Method Start writing out decimal expansions:

$x = 1.4142\,34567\ldots.$

$y = 1.4144\,56789\ldots.$

Find 1st place they differ, split difference:

$$r = 1.4143 = \frac{14143}{10000}$$

Also....

Thm 3.3.15 $\forall\, x, y \in \mathbb{R},\ x<y\ \exists$ irrational $z \in \mathbb{R} \setminus \mathbb{Q}$ such that $x < z < y$.

(irrat'ls are dense in $\mathbb{R}$).

Pf: By density of $\mathbb{Q}$, $\exists\, r \in \mathbb{Q}$ s.t.

$$\frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}}$$

yes if $r \neq 0$ (which is doable)

$$\Rightarrow\ x < \underbrace{r \cdot \sqrt{2}} < y$$

(rat'l)(irr'l) always irrat'l?