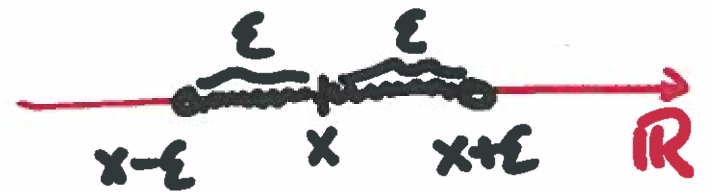
§ 3.4 Topology of $\mathbb{R}$

In this context, "topology" refers to open and closet sets in $\mathbb{R}$.

With sequences, limits we talk about "points close to x". This section lays groundwork for putting "close to" on a rigorous footing:

- * neighborhoods
- * interior pts, boundary pts
- * open, closed sets in $\mathbb{R}$
- accumulation pts, closure

Def Let $x \in \mathbb{R}$, $\varepsilon > 0$. Then



$$N(x;\varepsilon) = \{ y \in \mathbb{R} \mid |y-x| < \varepsilon \} = \{ y \in \mathbb{R} \mid |x-y| < \varepsilon \}$$

is the neighborhood (nbhd) centered at x with radius ε. Also $\exists$ "punctured" nbhd:

$$N^*(x;\varepsilon) = \{ y \in \mathbb{R} \mid 0 < |y-x| < \varepsilon \}$$

Equivalently

$$N(x;\varepsilon) = (x-\varepsilon, x+\varepsilon)$$

$$N^*(x;\varepsilon) = (x-\varepsilon, x) \cup (x, x+\varepsilon)$$

$N(x;\varepsilon) = \{\, |y-x| < \varepsilon \,\}$ $\quad$ $N^*(x;\varepsilon) = \{\, 0 < |y-x| < \varepsilon \,\}$

<u>Ex</u> $N(2;1) = (1,3)$

$N(0;1/2) = (-1/2, 1/2)$

$N^*(10;3) = (7,10) \cup (10,13)$

<u>Def</u> $x \in S \subseteq \mathbb{R}$ is an <u>interior point</u> of S if $\exists\, \varepsilon > 0$ such that $N(x;\varepsilon) \subseteq S$.

$N(1;4) = (-3,5) \not\subseteq S$, but $N(1;1/2) \subseteq S$, so 1 is int. pt of S.

<u>Ex</u> $(0,3)$

3 <u>not</u> int pt. Not in $S = (0,3)$, and <u>any</u> nbhd of 3 includes pts not in S.

Conversely, if every nbhd of x contains pts in S ($N \cap S \neq \phi$) and also contains pts not in S ($N \cap S^c \neq \phi$) then x is a boundary point of S.

Ex 3 is bdy pt of $(0,3)$.

Every $N(3;\varepsilon) = (3-\varepsilon, 3+\varepsilon)$ includes pts in $(0,3)$ and not in $(0,3)$.

Def int S = set of all interior pts of S

bd S = set of all bdy pts of S.

<u>Ex</u> $S=\{0,2,4\}$

<u>no</u> int pts, $\operatorname{int} S=\emptyset$

$0\in \operatorname{bd} S$, b/c any $N(0;\varepsilon)$ contains pts not in S (neg #s) and pt (0) in S. $2,4\in \operatorname{bd} S$.

$T=[0,1)$

$\operatorname{int} T=(0,1)$

$\operatorname{bd} T=\{0,1\}$

⚠ int pts must be in the set; bdy pts need not be!

Open/Closed Sets

⚠ Our approach slightly different than the book's; everything will work out the same, but our def's are the books thm's and vice-versa.

Def $S \subseteq \mathbb{R}$ is open in $\mathbb{R}$ if every $x \in S$ is an interior point.

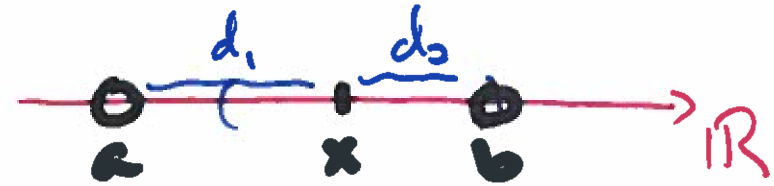
- equivalently, $S \subseteq \text{int } S$.
- Since $\text{int } S \subseteq S$, could also say $S = \text{int } S$.

Examples

① Any interval $(a,b) = \{a < x < b\}$ is open.

Let $\varepsilon = \min\{d_1, d_2\}$

$= \min\{|x-a|, |x-b|\} \Rightarrow N(x;\varepsilon) \subseteq (a,b).$



② Since $N(x;\varepsilon) = (x-\varepsilon, x+\varepsilon)$ (an open interval),

by part ①, $N(x;\varepsilon)$ is open

"open nbhds"

③ $\mathbb{R}$ Let $x \in \mathbb{R}$. <u>Any</u> nbhd $N(x;\varepsilon) \subseteq \mathbb{R}$

so $\mathbb{R}$ is open.

④ $\emptyset \subseteq \mathbb{R}$ open for "trivial" reasons:

if $x \in \emptyset$ then x is int pt of $\emptyset$.

always F, hence implication is T.

⑤ $S = (0,1) \cup (3,4)$

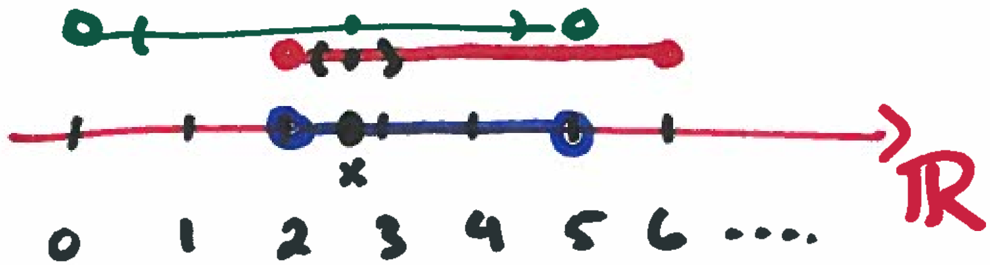
Let $x \in S$, so $x \in (0,1)$ or $x \in (3,4)$

If $x \in (0,1)$, which is an open set, then $\exists$ $N(x;\varepsilon) \subseteq (0,1) \subseteq S$.

If $x \in (3,4)$, then because $(3,4)$ is open, $\exists\, \varepsilon > 0$ s.t. $N(x;\varepsilon) \subseteq (3,4) \subseteq S$.

⑥ $S = (0,5) \cap (2,6)$

[$= (2,5)$, open]



Let $\underset{\sim}{x \in S} = (0,5) \cap (2,6)$. Then

$x \in \cancel{S}(0,5)$ <u>and</u> $x \in (2,6)$.

Because $x \in (0,5)$, $\exists\ \varepsilon_1 > 0$ s.t. $N(x;\varepsilon_1) \subseteq (0,5)$.

Because $x \in (2,6)$, $\exists\ \varepsilon_2 > 0$ s.t. $N(x;\varepsilon_2) \subseteq (2,6)$

<u>Key</u> if $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$, then

$N(x;\varepsilon) \subseteq S$. Hence S is open

⚠ $S = (0,1) \cap (3,4) = \emptyset$ which is still open.

More generally...

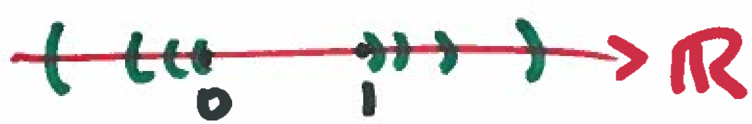
Thm 3.4.10

(a) Any union of open sets is open.
↳ finitely or infinitely many (!!)

(b) An intersection of finitely many open sets is open.

Ex $\bigcup_{n \in \mathbb{N}} (0,n) = (0,1) \cup (0,2) \cup (0,3) \cup \cdots = (0,\infty)$ open

$\bigcap_{n \in \mathbb{N}} \left(-\frac{1}{n}, 1+\frac{1}{n}\right) = (-1,2) \cap \left(-\frac{1}{2}, \frac{3}{2}\right) \cap \left(-\frac{1}{3}, \frac{4}{3}\right) \cap \cdots = [0,1]$



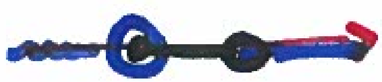
NOT open

Def $S \subseteq \mathbb{R}$ is closed if $\mathbb{R} \setminus S = S^c$ is open.

⚠ closed, open NOT opposites.

$[0,1)$ is neither:

not open because 0 not int. point.



not closed b/c $[0,1)^c = \underbrace{(-\infty,0) \cup [1,\infty)}_{\text{not open}}$

(1 is not int. pt)

A Topologist "Opens" a Store....

Examples

① $S = [0,1]$ closed b/c $[0,1]^c = \mathbb{R} \setminus [0,1]$

$= \underbrace{(-\infty, 0) \cup (1, \infty)}_{\text{union of open sets is open.}}$

S^c open $\Rightarrow S$ closed.

② $S = \emptyset$. $S^c = \emptyset^c = \mathbb{R} \setminus \emptyset = \mathbb{R}$,

which is open - hence $\emptyset$ is closed.

③ $\mathbb{R}$. $\mathbb{R}^c = \mathbb{R} \setminus \mathbb{R} = \emptyset$ open, which means

$\mathbb{R}$ is closed.

$\emptyset$, $\mathbb{R}$ both open, closed : they are clopen.

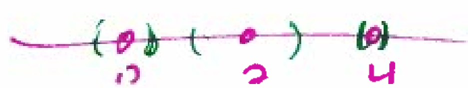
$\exists$ a different characterization.

Thm $S \subseteq \mathbb{R}$ is closed iff it contains all its bdy pts ($\operatorname{bd} S \subseteq S$).

Ex $[a,b]$ closed b/c $[a,b]^c = (-\infty, a) \cup (b, \infty)$ open.

$\operatorname{bd}[a,b] = \{a,b\} \subseteq [a,b]$

$\{0,2,4\}$. $\operatorname{bd}\{0,2,4\} = \{0,2,4\} \subseteq \{0,2,4\}$



Thm (a) Any (finite or infinite) intersection of closed sets is closed.

(b) Any finite union of closed sets is closed.

Pf (b) Suppose $A_1, A_2, \ldots, A_n$ are closed.

$\bigcup_{k=1}^{n} A_k$ is closed if its comp. is open.

$$\underbrace{\left(\bigcup_{k=1}^{n} A_k\right)^c =}_{\text{De Morgan's Laws}} \bigcap_{k=1}^{n} \underbrace{(A_k^c)}_{\text{open}}$$

is open b/c its finite $\cap$ of open sets.

How about $A := \{ \frac{1}{n} : n \in \mathbb{N} \} = \{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots \}$

<u>Claim</u> every $a \in A$ is a bdy pt. Any nbhd [redacted] ct'd at $\frac{1}{n}$ will include pts not in A.

<u>Claim</u> 0 <u>also</u> a bdy pt. Any Nbhd $N(0;\varepsilon)$ includes pts <u>not</u> in A (like neg. #'s) but also includes pts in <u>A</u>.

$\forall\ \varepsilon > 0,\ \exists n \ni 0 < \frac{1}{n} < \varepsilon$, so $\frac{1}{n} \in N(0;\varepsilon)$

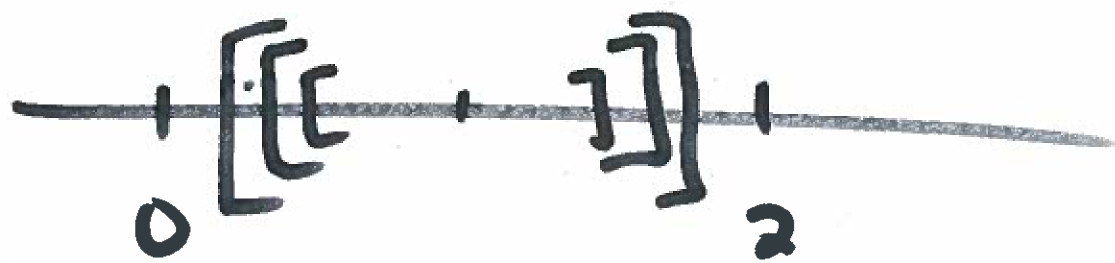
$\Rightarrow$ A <u>not</u> closed. (doesn't include all its bdy pts)

Inf. $\cup$'s of closed sets can fail to be closed.

⚠ $\bigcup [-n,n] = \cdots = \mathbb{R}$ open but still closed.

- $\bigcup_{n\in\mathbb{N}} \left[\frac{1}{n}, 2-\frac{1}{n}\right] = [1,1]\cup\left[\frac{1}{2},\frac{3}{2}\right]\cup\left[\frac{1}{3},\frac{5}{3}\right]$

$= (0,2)$



Not in course, but read if you're interested

$\exists$ notion of "closure" of a set.

Also a hybrid of int/bd pts called "accumulation" pts.

Also, § 3.5 Compact Sets not in course but....

Def A set $S \subseteq \mathbb{R}$ is compact if it is closed and bounded (above and below)

(Not actually defn - this is Heine-Borel Thm).