§4.1 Sequences and Convergence

Why study sequences? Basic math'l object.
Much of calculus can be described/developed w/ seq's (and series): continuous fns, limits (deriv's, integrals)

Informally, a sequence is a list of numbers:

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots$$

Formally A seq. in $\mathbb{R}$ is a fn $f : \mathbb{N} \to \mathbb{R}$, where $f(i)$ is i^{th} # in list. So above,

$f(n) = \frac{1}{n}$ $f(1) = 1$ $f(3) = 1/4$

$f(2) = 1/2$ $\vdots$

Usually we avoid the fn notation and use subscripts: $a_1 = f(1)$, $a_2 = f(2)$, $a_3 = f(3), \ldots$

For above, $(a_n) = \left(\frac{1}{n}\right)$

$(a_n) = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right)$

~~So for above sequence, write:~~

⚠ (a_n) = the sequence $(a_1, a_2, a_3, a_4, \dots)$

$\{a_n\}$ = the set of #'s in the sequence.

<u>Ex</u> $a_n = \sin\left(\frac{\pi}{2} \cdot n\right)$

$(a_n) = (1, 0, -1, 0, 1, 0, -1, 0, \dots)$

$\{a_n\} = \{-1, 0, 1\}$

⚠ many, many books use $\{a_n\}$ for what our book calls (a_n).

<u>Ways to Define/Express a Sequence</u>

① Give a formula to describe the n^{th} term

$a_n = \frac{1}{n}$ <u>OR</u> $(a_n) = \left(\frac{1}{n}\right) = \left(\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \dots\right)$

② Give 1^{st} term(s) and recursive formula

$a_1 = 1$, $a_n = 2a_{n-1} + 1$ gives $(1, 3, 7, 15, \ldots)$

$a_1 = a_2 = 1$ $a_n = a_{n-1} + a_{n-2}$ $(1, 1, 2, 3, 5, 8, 13, \ldots)$

③ List enough terms to establish a pattern.

$(a_n) = (1, 4, 9, 16, 25, 36, \ldots)$ (square #'s)

⚠ Risky! what if somebody doesn't see pattern? Or a different pattern?

$b_4 = 63$.

$(b_n) = (0, 7, 26, \ldots)$ $[b_n = n^3 - 1]$

$(c_n) = (1, 11, 21, 1211, 111221, \ldots)$

Search: **seq:0,7,26**

Displaying 1-10 of 11 results found. page 1 2

Sort: relevance | references | number | modified | created Format: long | short | data

A049453 Second pentagonal numbers with even index: n*(6*n+1). +20 17

0, 7, 26, 57, 100, 155, 222, 301, 392, 495, 610, 737, 876, 1027, 1190, 1365, 1552, 1751, 1962, 2185, 2420, 2667, 2926, 3197, 3480, 3775, 4082, 4401, 4732, 5075, 5430, 5797, 6176, 6567, 6970, 7385, 7812, 8251, 8702, 9165, 9640, 10127, 10626 (list; graph; refs; listen; history; text; internal format)

OFFSET	0,2
COMMENTS	Number of edges in the join of the complete tripartite graph of order 3n and the cycle graph of order n, K_n,n,n * C_n - Roberto E. Martinez II, Jan 07 2002 Sequence found by reading the line (one of the diagonal axes) from 0, in the direction 0, 7,..., in the square spiral whose vertices are the generalized pentagonal numbers A001318. - Omar E. Pol, Sep 08 2011 First bisection of A036498. - Bruno Berselli, Nov 25 2012
LINKS	Table of n, a(n) for n=0..42.
FORMULA	G.f.: x*(7+5*x)/(1-x)^3. a(n) = 12*n+a(n-1)-5 with n>0, a(0)=0. - Vincenzo Librandi, Aug 06 2010
MAPLE	seq(binomial(6*n+1, 2)/3, n=0..42); - Zerinvary Lajos, Jan 21 2007
MATHEMATICA	s=0; lst={s}; Do[s+=n++ +7; AppendTo[lst, s], {n, 0, 7!, 12}]; lst [From Vladimir Joseph Stephan Orlovsky, Nov 16 2008]
CROSSREFS	Cf. A001318, A005449, A033568, A033570, A036498, A049452, A185019, A194454.
KEYWORD	nonn,easy
AUTHOR	Joe Keane (jgk(AT)jgk.org).
STATUS	

approved

A068601 n^3-1. +20 17

0, 7, 26, 63, 124, 215, 342, 511, 728, 999, 1330, 1727, 2196, 2743, 3374, 4095, 4912, 5831, 6858, 7999, 9260, 10647, 12166, 13823, 15624, 17575, 19682, 21951, 24388, 26999, 29790, 32767, 35936, 39303, 42874, 46655, 50652, 54871, 59318, 63999 (list; graph; refs; listen; history; text; internal format)

OFFSET 1,2

COMMENTS a(n) is the least positive integer k such that k can only contain 'n-1' in exactly 2 different bases B, where 1<B<=k.

A129294(n) = number of divisors of a(n) that are not greater than n. - Reinhard Zumkeller, Apr 09 2007

Apart from the first term, the same as A135300. R. J. Mathar, Apr 29 2008

A058895(n)^3 +a(n)^3 +A033562(n)^3 = A185065(n)^3. _ Vincenzo Librandi, Mar 13 2012

Numbers n such that for every nonnegative integer m, n^(3*m+1) + n^(3*m) is a cube. - Arkadiusz Wesolowski, Aug 10 2013

LINKS Nathaniel Johnston, Table of n, a(n) for n = 1..10000

Index to sequences with linear recurrences with constant coefficients, signature (4,-6,4,-1).

FORMULA Partial sums of A003215, hex (or centered hexagonal) numbers: 3n(n+1)+1. - Jonathan Vos Post, Mar 16 2006

G.f.: x^2*(7-2*x+x^2)/(1-x)^4. [Colin Barker, Feb 12 2012]

EXAMPLE For n=6; 215 written in bases 6 and 42 is 555, 55 and (555, 55) are exactly 2 different bases.

MATHEMATICA f[n_]:=n^3-1; f[Range[60]] (*From Vladimir Joseph Stephan Orlovsky, Feb 14 2011*)

LinearRecurrence[{4, -6, 4, -1}, {0, 7, 26, 63}, 50]] (* Vincenzo Librandi, Mar 11 2012 *)

PROG (PARI) a(n)=n^3-1

(MAGMA) [n^3-1: n in [1..40]]; // Vincenzo Librandi, Mar 11 2012

CROSSREFS Cf. A000217, A005448, A016921, A058895, A033562, A185065.

KEYWORD easy,nonn

AUTHOR Naohiro Nomoto, Mar 28 2002

STATUS approved

A219695 For odd numbers 2n - 1, half the difference between the largest divisor not exceeding the square root, and the least divisor not less than the square root. +20 4

0, 1, 2, 3, 0, 5, 6, 1, 8, 9, 2, 11, 0, 3, 14, 15, 4, 1, 18, 5, 20, 21, 2, 23, **0, 7, 26,** 3, 8, 29, 30, 1, 4, 33, 10, 35, 36, 5, 2, 39, 0, 41, 6, 13, 44, 3, 14, 7, 48, 1, 50, 51, 4, 53, 54, 17, 56, 9, 2, 5, 0, 19, 10, 63, 20, 65, 6, 3, 68, 69, 22, 1, 12, 7, 74, login

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Integer Sequences

1,11,21,1211 Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

Search: **seq:1,11,21,1211**

Displaying 1-5 of 5 results found. page 1

Sort: relevance | references | number | modified | created Format: long | short | data

A005150 Look and Say sequence: describe the previous term! (method A - initial term is 1). +20
(Formerly M4780) 117

1, 11, 21, 1211, 111221, 312211, 13112221, 1113213211, 31131211131221, 13211311123113112211, 11131221133112132113212221, 3113112221232112111312211312113211 (list; graph; refs; listen; history; text; internal format)

OFFSET 1,2

COMMENTS

Method A = 'frequency' followed by 'digit'-indication.

Also known as the "Say What You see" sequence.

Only the digits 1, 2 and 3 appear in any term. - Robert G. Wilson v Jan 22 2004.

All terms end by 1 (the seed) and, except the third a(3), begin by 1 or 3. [Jean-Christophe Hervé, May 07 2013]

Proof that 333 never appears in any a(n): suppose it appears for the first time in a(n); because of 'three 3' in 333, it would imply that 333 is also in a(n-1), which is a contradiction. [Jean-Christophe Hervé, May 09 2013]

REFERENCES

J. H. Conway, The weird and wonderful chemistry of audioactive decay, Eureka 46 (1986) 5-16.

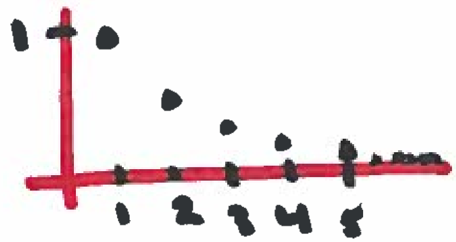
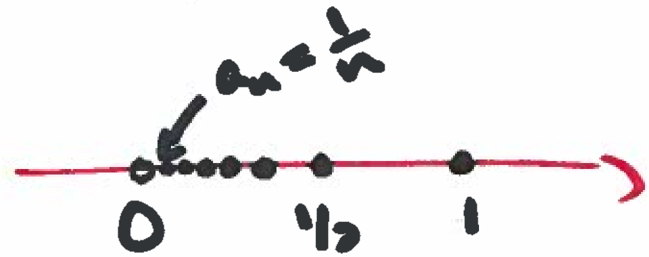
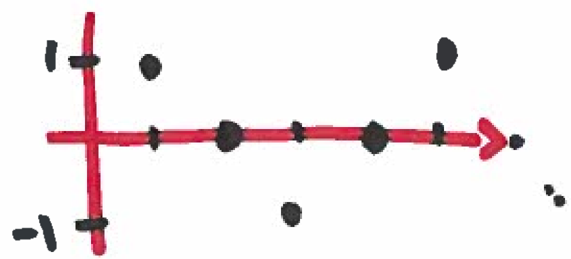
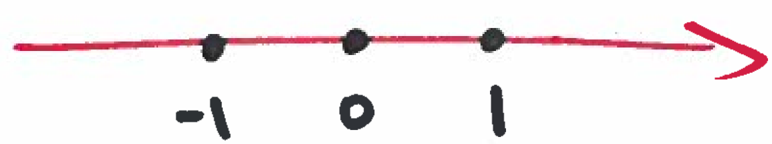
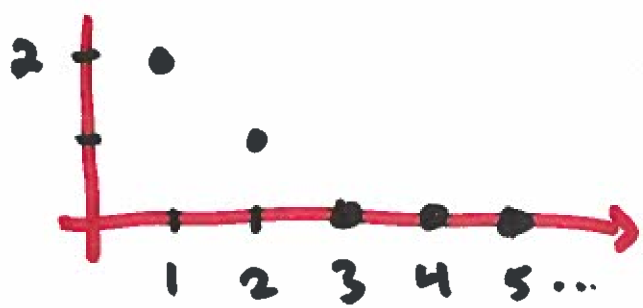
J. H. Conway, The weird and wonderful chemistry of audioactive decay, in T. M. Cover and Gopinath, eds., Open Problems in Communication and Computation, Springer, NY 1987, pp. 173-188.

S. B. Ekhad and D. Zeilberger, Proof of Conway's lost cosmological theorem, Elect. Res. Announcements Amer, Math. Soc., 3 (1997), 78-82.

S. Eliahou and M. J. Erickson, Mutually describing multisets and integer partitions, Discrete Mathematics, Volume 313, Issue 4, 28 February 2013,

(4) Graphically (two common ways)

Helpful, but not rigorous...

	$\mathbb{N} \times \mathbb{R}$	$\mathbb{R}$
$a_n = \frac{1}{n}$		
$b_n = \sin\left(\frac{\pi}{2} n\right)$		
$c_n = \lfloor \frac{2}{n} \rfloor$		

Def (s_n) converges to $s \in \mathbb{R}$, written

$$\lim_{n\to\infty} s_n = s \quad \text{OR} \quad \lim s_n = s \quad \text{OR} \quad s_n \to s$$

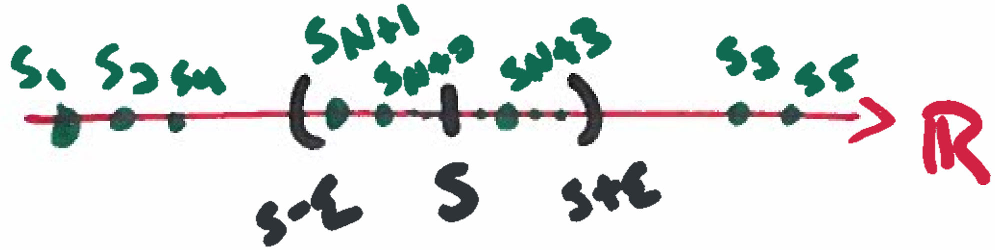
$$\forall\, \varepsilon>0 \;\; \exists\, N \in \mathbb{N} \text{ (or } \mathbb{R}) \text{ s.t. } n>N \Rightarrow |s_n - s| < \varepsilon.$$

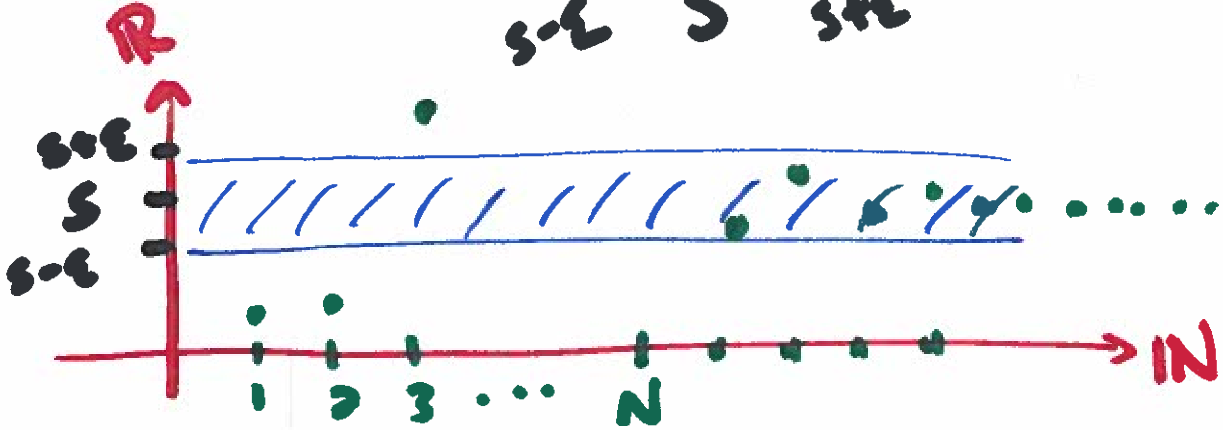
In words: $s_n \to s$ if eventually every s_n close to s

$n > N$, some N $\qquad\qquad$ $|s_n - s| < \varepsilon$

⚠ Order matters! You don't choose ε. It's given to you, and your challenge is to prove you can (eventually) make s_n be within ε of s.

Visually





In general, to prove $s_n \to s$,

Step 1: "Think" (prepare, algebra)

Do algebra to figure out how large n has to be to ensure $|s_n - s| < \varepsilon$. Set N equal to this value.

Step 2: "Proof." Do algebra in reverse.

Given $\varepsilon > 0$, choose $N =$ (magic #, depends on ε)

Then $\forall\, n > N$, we have $|s_n - s| =$ algebra $< \varepsilon$

$\underline{\text{Ex}}$ $(s_n) = \left(\frac{n-1}{n}\right) = \left(0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots\right)$

We suspect $\lim s_n = 1$, i.e. $s_n \to 1$.

<u>Preparation</u>

Want: $|s_n - s| < \varepsilon$, i.e. $\left|\frac{n-1}{n} - 1\right| < \varepsilon$

$$-\varepsilon < \frac{n-1}{n} - 1 < \varepsilon$$

$$-\varepsilon < \frac{\cancel{n} - 1 - \cancel{n}}{n} < \varepsilon$$

$$\underbrace{-\varepsilon < -\frac{1}{n}} < \varepsilon$$

$\frac{1}{n} < \varepsilon$, so $n > 1/\varepsilon$.

no useful information.

Proof that $\frac{n-1}{n} \to 1$

$\Leftrightarrow \varepsilon = \frac{1}{N}$

Given $\varepsilon > 0$, set $N = 1/\varepsilon$. Then $n > N$ forces

$$
\begin{aligned}
|s_n - s| &= \left| \frac{n-1}{n} - 1 \right| \\
&= \left| \frac{n-1}{n} - \frac{n}{n} \right| \\
&= \left| \frac{n-1-n}{n} \right| \\
&= \left| -\frac{1}{n} \right| \\
&= \frac{1}{n} < \varepsilon \quad \left(\text{because } n > \frac{1}{\varepsilon}\right).
\end{aligned}
$$

Note: $N \in \mathbb{R}$ here, not $\mathbb{N}$. ok b/c could use arch. princ. to come up with nat'l # $> \frac{1}{\varepsilon}$.

Ex 4.1.6 Show $s_n = \frac{n^2+2n}{n^3-5} \to 0$

Algebra/Preparation step

Need to find condition on s such that

$$\left| \frac{n^2+2n}{n^3-5} - 0 \right| < \varepsilon.$$

Tricks of the trade: find "nicer" sequence which $\to 0$ and is bigger ~~that~~ than ours:

$$\left| \frac{n^2+2n}{n^3-5} - 0 \right| \leq \left(\qquad \right) < \varepsilon$$

Showing $\frac{n^2+2n}{n^3-5} \to 0$

① We only care about what <u>eventually</u> happens, so we can assume $\boxed{n \geq 2}$ so that

$n^2+2n \geq 0$ and $n^3-5 \geq 0$

$$\Rightarrow \left|\frac{n^2+2n}{n^3-5}\right| = \frac{n^2+2n}{n^3-5}.$$

② Try to find a simpler (larger) seq. which bounds ours. <u>KEY</u>

$$\frac{n^2+2n}{n^3-5} < \frac{p(n)}{q(n)} \quad \text{if} \quad \begin{array}{l} p(n) > n^2+2n \\ q(n) < n^3-5 \end{array}$$

Helps if p, q have form $k \cdot n^{\text{power}}$, degree$(q) >$ deg(p)

Showing $\dfrac{n^2+2n}{n^3-5} \longrightarrow 0$

- For large n (specifically, $\underline{n \geq 3}$)

  $n^2 > 2n \implies n^2+n^2 > n^2+2n$

  $\underbrace{2n^2}_{p(n).} > n^2+2n$

- For large n (specifically, $\underline{n \geq 3}$),

  $n^3-5 > n^3$ never true!

  How about $\frac{1}{2}n^3$?

  $\underbrace{n^3-5 > \frac{1}{2}n^3}_{\text{solve}} \implies \frac{1}{2}n^3 > 5$

  $n^3 > 10$

  $n > \sqrt[3]{10} \approx 2.15\ldots$

  $n^3-5 > \underbrace{\frac{1}{2}n^3}_{q(n).}$

Showing $\dfrac{n^3+2n}{n^3-5} \longrightarrow 0$

So for $n \geq 3$, we know $\dfrac{n^3+2n}{n^3-5} < \dfrac{2n^2}{\frac{1}{2}n^3} = \dfrac{4}{n}$.

③ Any N for simpler sequence $\left(\frac{4}{n}\right)$ will work for our messy one, too.

Want: $\left|\frac{4}{n} - 0\right| = \frac{4}{n} < \varepsilon \iff n > 4/\varepsilon$.

So if $n > \frac{4}{\varepsilon}$ **and** $n \geq 3$

$$\underbrace{\frac{n^3+2n}{n^3-5} <}_{n \geq 3} \underbrace{\frac{4}{n} < \varepsilon}_{n > 4/\varepsilon}.$$

<u>Proof</u> that $\frac{n^2+2n}{n^3-5} \to 0$ (Finally - whew!)

Given $\varepsilon>0$, set $N = \max\{3, 4/\varepsilon\}$. Then $n>N$

$$\Rightarrow |s_n - s| = \left|\frac{n^2+2n}{n^3-5} - 0\right| \underset{n\geq 2}{=} \frac{n^2+2n}{n^3-5} \underset{n\geq 3}{<} \frac{4}{n} \underset{n>\frac{4}{\varepsilon}}{<} \varepsilon$$

This is all a special case of

<u>Thm</u> Suppose $a_n \to 0$. If for $k>0$, $m\in\mathbb{N}$ we have $|s_n - s| \leq k\cdot a_n$ for all $n>m$. then $s_n \to s$.

<u>Notes</u> ① $\left|\frac{n^2+2n}{n^3-5} - 0\right| \leq 4\left(\frac{1}{n}\right)$

s_n s k a_n $(m=3)$

② This is the Squeeze Thm!

If $0 \le f(x) \le h(x)$

and $\lim_{n\to\infty} 0 \le \lim_{n\to\infty} f(x) \le \lim_{n\to\infty} h(x) = 0$

$$0 \le \lim f(x) \le 0$$

$$\Rightarrow \lim f(x) = 0.$$

Pf Given $\varepsilon > 0$, need to show $|s_n - s| < \varepsilon$ for some eventual values of n.

Because $a_n \to 0$, $\exists\, N_1$ s.t. $\overset{|a_n - 0|}{\overset{\shortparallel}{|a_n|}} < \frac{\varepsilon}{k}$

Let $N = \max\{m, N_1\}$ Then $n > N$

$$\Rightarrow |s_n - s| \underset{n>m}{\le} k|a_n| \underset{n>N_1}{<} k(\varepsilon/k) = \varepsilon$$

Q: How to prove a sequence diverges?

Recall the def^n:

$s_n \to s$ iff $\forall \varepsilon > 0 \ \exists N \ni \overset{\text{implied } \forall}{n} > N \Rightarrow |s_n - s| < \varepsilon$.

$s_n \not\to s$ iff $\exists \varepsilon > 0 \ni \forall N \ \exists n \ni n > N$ and $|s_n - s| \geq \varepsilon$

i.e. $\exists \varepsilon > 0$ for which you can't ever guarantee s_n is within ε of s!

Ex $\frac{1}{n} \not\to 9$ b/c we can't guarantee $\frac{1}{n}$ will be with ε units of 9 for $\varepsilon = 1$. In fact, $|\frac{1}{n} - 9| \geq \varepsilon = 1 \ \forall n$ – not even a matter of finding the n!

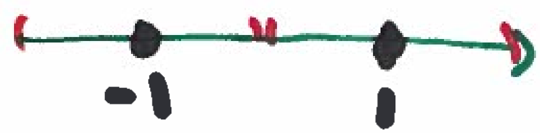
So to prove s_n **DIVERGES**, you must show it cannot converge to *any* $s \in \mathbb{R}$ – that's hard!

Must exhibit some $\varepsilon > 0$ s.t. no matter *how* large n gets, you cannot guarantee $|s_n - s| < \varepsilon$ for any s.

Ex Prove $s_n = (-1)^n$ diverges. $(s_n) = (-1, 1, -1, 1, -1, 1, \dots)$

Let $\varepsilon = 1/2$. suppose $s_n \to s$.

$\exists N$ s.t. $n > N$, $|s_n - s| < \varepsilon = 1/2$.



Thus $-\frac{1}{2} < s_n - s < \frac{1}{2}$.

No matter how large n is, s_n still oscillates b/w $-1, 1$.

<u>n odd</u> $\Rightarrow s_n = -1$, so $-\frac{1}{2} < -1 - s < \frac{1}{2} \Rightarrow -\frac{3}{2} < s < -\frac{1}{2}$

<u>n even</u> $\Rightarrow s_n = 1$, so $-\frac{1}{2} < 1 - s < \frac{1}{2} \Rightarrow \frac{1}{2} < s < \frac{3}{2}$

Thus $-\frac{3}{2} < s < -\frac{1}{2} < \frac{1}{2} < s < \frac{3}{2}$

This is a contradiction. Hence s_n does not converge to <u>any</u> s.

$\underline{\text{Thm}}$ If $s_n \to s$, $s_n \to t$ then $s = t$. (limits are unique)

$\underline{\text{Pf}}$ We want to show $s = t \iff s - t = 0$

$\iff |s-t| = 0$

(Thm 3.2.8, followup HW) $\iff |s-t| < \varepsilon \ \forall \varepsilon > 0.$

Since $s_n \to s$, $s_n \to t$, given $\varepsilon > 0$, we know

$\exists M$ s.t. $n > M \Rightarrow |s_n - s| < \varepsilon$

$\exists \underline{K}$ s.t. $n > \underline{K} \Rightarrow |s_n - \underline{t}| < \varepsilon$

(*) Could use $\varepsilon/2$ for these

Then $N = \max\{M, K\} \Rightarrow$ both ineq's are true.

For $\underline{n > N}$ we therefore have

$$|s-t| = |(s_n - t) - (s_n - s)| \quad \text{(add 0)}$$

$$\le |s_n - t| + |s_n - s| \quad (\triangle \text{ ineq})$$

$$< \varepsilon + \varepsilon \qquad * \ \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

So $|s-t| < 2\varepsilon$. $*$ ε.

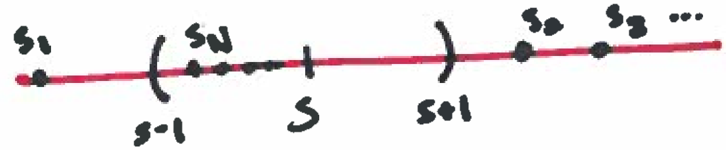
as desired.

$*$ Idea could go back and find M, K such that $|s_n - s|$, $|s_n - t|$ are less than $\varepsilon/2$. Then $|s-t| \le |s_n - s| + |s_n - t|$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thm A convergent sequence is bounded

(Read the proof of Thm 4.1.13 in book – have a pencil and scratch paper in hand to draw a number line and follow along:



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