§4.2 Limit Theorems

A section full of time-saving tools!

Goal Recover what we "know" about seq's and limits from calculus – i.e. show that they follow from our rigorous def^n involving ε, N....

<u>Thm 4.2.1</u> Suppose $s_n \to s$, $t_n \to t$. Then

(a) $\lim (s_n + t_n) = s + t = \lim s_n + \lim t_n$ $[s_n + t_n \to s+t]$

(b) $\lim (k s_n) = ks = k \cdot \lim s_n$ $[k s_n \to ks]$

$\lim (k + s_n) = k + s = k + \lim s_n$ $[k + s_n \to k+s]$

$\forall k$ in $\mathbb{R}$.

(c) $\lim (s_n t_n) = s \cdot t = (\lim s_n)(\lim t_n)$ etc....

(d) $\lim (s_n / t_n) = \dfrac{s}{t} = \dfrac{\lim s_n}{\lim t_n}$

provided that $t_n \neq 0$ $\forall n$ and $t \neq 0$.

(Not strictly necessary - see exercises)

These are (baby versions of) <u>limit laws</u>:

$$\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x),$$ provided those limits exist.

Thm 4.2.1 Makes life much easier!

b/c these limits do not exist.

Ex $$\lim \frac{4n^2-3}{5n^2-2n} \neq \frac{\lim 4n^2-3}{\lim 5n^2-2n}$$

(algebra) $\parallel$

$$\lim \frac{4-3/n^2}{5-2/n} \overset{4.2.1\,(d)}{=} \frac{\lim 4-3/n^2}{\lim 5-2/n}$$

$$\overset{(a)}{=} \frac{\lim 4 + \lim (-1)3/n^2}{\lim 5 + \lim (-1)2/n}$$

$$\overset{(b)}{=} \frac{\lim 4 + (-1)\lim 3/n^2}{\lim 5 + (-1)\lim 2/n}$$

$$= \ldots = \frac{4+0}{5+0} = 4/5.$$

For these prob's we can assume

$$\frac{c}{n^k} \longrightarrow 0$$

for $c \in \mathbb{R}$, $k>0$

Ex Prove $\frac{n^2+2n}{n^3-5} \longrightarrow 0$

$$\lim \frac{n^2+2n}{n^3-5} = \lim \frac{(n^2+2n)\frac{1}{n^3}}{(n^3-5)\frac{1}{n^3}} = \lim \frac{\frac{1}{n}+\frac{2}{n^2}}{1-5/n^3}$$

$$= \frac{\lim \frac{1}{n}+\frac{2}{n^2}}{\lim 1-\frac{5}{n^3}} = \frac{0+0}{1-0} = 0.$$

Remember: when we apply limit laws, must be sure the "new" limits actually exist.

(Some examples will work backwards to avoid this.)

Pf of (a), $[s_n \to s \text{ and } t_n \to t] \Rightarrow (s_n + t_n) \to s+t$

Let $\varepsilon > 0$. We need to show we can make

$$|(s_n + t_n) - (s+t)| < \varepsilon.$$

Using $\triangle$ ineq,

$$|(s_n - s) + (t_n - t)| \le |s_n - s| + |t_n - t|$$

Because $s_n \to s$, $t_n \to t$, we can find

(*) N_1 such that $n > N_1$, $|s_n - s| < \varepsilon/2$

N_2 such that $n > N_2$, $|t_n - t| < \varepsilon/2$

If $n > \max\{N_1, N_2\}$ $(= N)$, then both ineq's true, so

$$|(s_n - s) + (t_n - t)| \le |s_n - s| + |t_n - t|$$

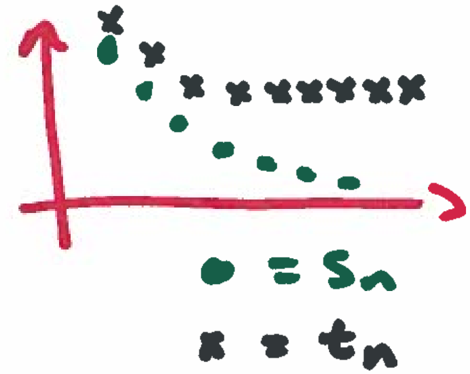
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

(*) N_1, N_2? M, K? N_s, N_t? No std answer...

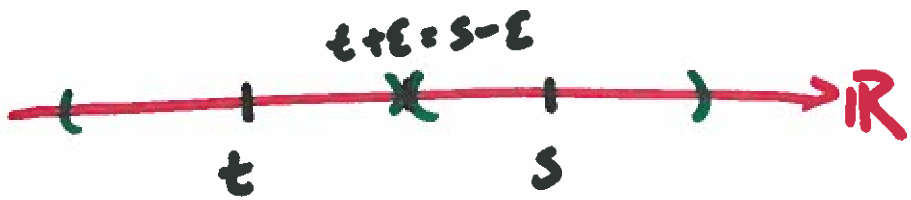
Thm 4.2.4 (Another "Squeeze" Thm)

Suppose $s_n \to s$, $t_n \to t$. If $s_n \le t_n \ \forall n \in \mathbb{N}$ then $s \le t$.

Pf Cool proof by contradiction.



Suppose not, that $t < s$



max of two constants, etc.

Choose $\varepsilon = (s-t)/2$. For large enough N, $n > N \Rightarrow$

$$s-\varepsilon < s_n < s+\varepsilon \qquad (|s_n - s| < \varepsilon)$$

$$t-\varepsilon < t_n < t+\varepsilon \qquad (|t_n - t| < \varepsilon)$$

$\Rightarrow t_n < t+\varepsilon = s-\varepsilon < s_n$ i.e. $t_n < s_n$ ↯.

<u>Corollary 1</u> Same setup and $0 \le s_n \le t_n$, $t_n \to 0 \Rightarrow s = 0$.

<u>Corollary 2</u> (Set $s_n = 0$, so $s = 0$) then $t \ge 0$.

What if, instead of $\frac{1}{n}$, $\frac{1}{n^2}$, $\frac{2n^3-3}{n^4+6}$, etc. we have seq's with factorials: $n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$

or exponential fns: e^n, 2^n, 3^n, ...

"Ratio Test for Sequences"

Thm 4.2.7 Suppose $s_n \geq 0$, ratios $\left(\frac{s_{n+1}}{s_n}\right)$ converge to L.
If $L < 1$ then $s_n \to 0$.

Think: "positive terms, getting smaller, so $s_n \to 0$"

" $\frac{s_{n+1}}{s_n} < 1$ "

Ex $a_n = \frac{1}{2^n}$ $\quad \frac{a_{n+1}}{a_n} = \frac{1/2^{n+1}}{1/2^n} = \frac{1}{2^{\cancel{n+1}}} \cdot \frac{\cancel{2^n}}{1} = \frac{1}{2} \longrightarrow \frac{1}{2} < 1.$

$\Rightarrow a_n \to 0$ by Thm 4.2.7.

$b_n = \frac{2^n}{n!}$ $\quad \frac{b_{n+1}}{b_n} = \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} = \frac{2^{n+1} \cdot \cancel{n(n-1)\cdots 2\cdot 1}}{2^n \cdot (n+1)\cancel{n(n-1)\cdots 2\cdot 1}}$

$= \frac{2}{n+1} \longrightarrow 0 < 1. \quad \Rightarrow b_n \to 0.$

$c_n = \frac{1}{n}$ $\quad \frac{c_{n+1}}{c_n} = \frac{1}{n+1} \cdot \frac{n}{1} = \frac{n}{n+1} \longrightarrow 1.$

⚠ Thm 4.2.7 tells us nothing about convergence of c_n. We know $c_n \to 0$, but must show that a different way.

Infinite Limits

$(s_n) = (n)$ diverges but for totally different reasons than $(t_n) = (1, 0, -1, 0, 1, 0, -1, 0, 1, \dots)$

It's useful to be able to describe this behavior:

s_n diverges to $+\infty$ ($\lim s_n = +\infty$, $s_n \to +\infty$) if: instead of getting close to a limit s, the numbers s_n eventually get larger than any given #.

$$\forall M \in \mathbb{R} \ \exists N \text{ s.t. } n > N \Rightarrow s_n > M.$$

Similarly $s_n \to -\infty$ (diverges!) if

$$\forall m \in \mathbb{R} \ \exists N \text{ s.t. } n > N \Rightarrow s_n < m.$$

You Read

Thm 4.2.12 Suppose $s_n \le t_n \ \forall n \in \mathbb{N}$

(a) $s_n \to +\infty \Rightarrow t_n \to \infty$

(b) $t_n \to -\infty \Rightarrow s_n \to -\infty$

Thm 4.2.13 $s_n > 0,\ s_n \to +\infty \Rightarrow \frac{1}{s_n} \to 0.$