Monotone Convergence Thm A monotone sequence is convergent iff it is bounded

VERY USEFUL because we can often prove seq converges w/ MCT without resorting to ε's, N's....

Ex $(a_n) = \left(1 - \frac{1}{n}\right) = \left(0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right)$

Claim a_n is bdd below by 0, above by 1.

i.e. $1 - \frac{1}{n} = \frac{n-1}{n} \geq 0 \ \forall n \in \mathbb{N}$, $\leq 1 \ \forall n \in \mathbb{N}$

(show work as in earlier sections)

Claim a_n incr'g: $a_{n+1} - a_n = \frac{n+1-1}{n+1} - \frac{n-1}{n} = \cdots = \frac{1}{n(n+1)} \geq 0$

$\Rightarrow$ a_n converges by MCT.

Ex 1 $a_n =$ (skip)

Ex 2 $s_1 = 1, \; s_{n+1} = \sqrt{1+s_n}$

$$(s_n) = \left(1, \; \sqrt{1+1}, \; \sqrt{1+\sqrt{2}}, \; \sqrt{1+\sqrt{1+\sqrt{2}}}, \; \ldots \right)$$

$1 \quad 1.414\ldots \quad 1.5537\ldots \quad 1.59805\ldots$

Claim 1 s_n incr'g. $s_2 = \sqrt{2} > 1 = s_1$; base case.
Now assume $s_k \geq s_{k-1}$. Then

$$s_{k+1} = \sqrt{1+s_k} \geq \sqrt{1+s_{k-1}} = s_k.$$ as desired.

Claim 2 $s_n \leq 2$. Base case: $a_1 = 1 \leq 2$. Now assume $s_k \leq 2$. Then $s_{k+1} = \sqrt{1+s_k} \leq \sqrt{1+2} = \sqrt{3} \leq 2$.

Since s_n is increasing, $s_n \geq s_1 = 1$ $\forall n$ so bdd below $\Rightarrow$ converges by MCT.

But what does $s_1=1$, $s_{n+1}=\sqrt{1+s_n}$ converge to?

KEY: for any sequence, $\lim_{n\to\infty} s_n = \lim_{n\to\infty} s_{n+1}$

Ex $(1, 1/2, 1/3, 1/4, 1/5, \ldots)$
$(1/2, 1/3, 1/4, 1/5, \ldots)$ } converge to same #.

Thus

$$\lim s_n = \lim s_{n+1}$$

$$\lim s_n = \lim \sqrt{1+s_n}$$ (suppose $s_n \to s$)

$$s = \sqrt{1+s}$$

$$s^2 = 1+s$$ ← introduce an extra sol'n.

$$s^2 - s - 1 = 0$$

Solve (get $s=\varphi$)

Cauchy Sequences

So far we've described "convergence" as elts of a sequence eventually bunching up next to a limit.
$\exists$ a different approach...

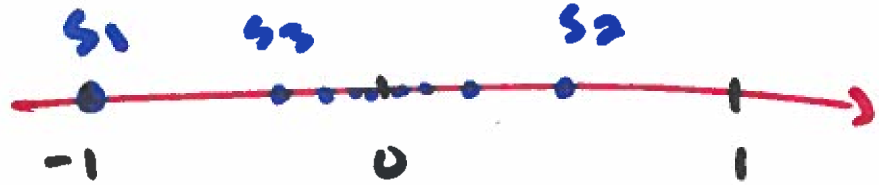
Def A sequence (s_n) of real numbers is a Cauchy Sequence if

$$\forall \varepsilon > 0 \ \exists N \text{ s.t. } n, m > N \Rightarrow |s_n - s_m| < \varepsilon.$$

i.e. eventually the #'s bunch up together

Cauchy: $\forall \varepsilon > 0 \ \exists N$ s.t. $n,m > N \Rightarrow |s_n - s_m| < \varepsilon$

Ex $s_n = \dfrac{(-1)^n}{n}$



dist of $5/6$, so if $\varepsilon = 5/6$, $n,m \geq 2$ forces

$|s_n - s_m| \leq \varepsilon$; $n,m > 2$

$\Rightarrow |s_n - s_m| < \varepsilon$

Given $\varepsilon > 0$, we know

$\exists N$ s.t. $\dfrac{1}{N} < \varepsilon/2$.

$\mathbb{N}$

Then $n,m > N$, we have

$$|s_n - s_m| \leq |s_n| + |s_m| = \left|\frac{(-1)^n}{n}\right| + \left|\frac{(-1)^m}{m}\right|$$

$$= \frac{1}{n} + \frac{1}{m}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\Rightarrow s_n$ is Cauchy.

Cauchy: $\forall \varepsilon > 0 \ \exists N$ s.t. $n, m > N \Rightarrow |s_n - s_m| < \varepsilon$

<u>Ex</u> $t_n = (-1)^n = \begin{cases} -1, & n \text{ odd} \\ 1, & n \text{ even} \end{cases}$

This is <u>not</u> a Cauchy sequence.

Can't force #'s in seq. to bunch up.

Say $\varepsilon = 1$. For any N, can always choose $n, m > N$ with n odd, m even

$\Rightarrow |s_n - s_m| = |(-1) - 1| = |-2| = 2 > \varepsilon$.

Cauchy: $\forall \varepsilon>0\ \exists N$ s.t. $n,m>N \Rightarrow |s_n-s_m|<\varepsilon$

Why do we care?

Thm (s_n) converges $\Longleftrightarrow$ (s_n) Cauchy.

Pf $\Leftarrow$ Not in this course.

$\Rightarrow$ Suppose $s_n \to s$, let $\varepsilon>0$ be given.

Must $\exists N$ s.t. $n,m>N. \Rightarrow |s_n-s_m|<\varepsilon$.

We know $\exists N$ s.t. $n>N \Rightarrow |s_n-s|<\varepsilon/2$.

Then $n,m>N$ gives

$$|s_n-s_m| = |s_n-s-(s_m-s)|$$
$$\le |s_n-s|+|s_m-s| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.$$