§4.3 Monotone and Cauchy Sequences

More ways to prove seq's converge without resorting to ε's and N's..... sort of.

Concepts: ★★★ 1. Monotone Convergence Thm (MCT)

★★ 2. ... Applications to certain recursive sequences.

★ 3. Cauchy Sequences

Def (s_n) is <u>increasing</u> if $s_n \le s_{n+1} \ \forall n$

<u>decreasing</u> if $s_n \ge s_{n+1} \ \forall n$

With $<, >$ instead of $\le, \ge$, we'd say

strictly incr'g, decr'g

Allowing for equality gives us monotonically incr'g, decr'g. (default)

(s_n) <u>monotone</u> $\iff$ (monotonically) incr'g or decr'g.

Examples of Seq's Which are....

Increasing

$a_n = n$, $b_n = n^2$ } strictly.

Decreasing

$c_n = \frac{1}{n}$, $d_n = -n$ } strictly.

Mon. incr'g, not strictly

$(s_n) = (22, 24, 24, \ldots\ldots)$

$(t_n) = (1,1,2,2,3,3,4,4,\ldots) = \left(\left\lfloor \frac{1}{2} + \frac{n}{2} \right\rfloor\right)$

Both $(a_n) = (1,1,1,1,1,1,\ldots.)$

Not Monotone $(b_n) = (1,-1,1,-1,1,-1,\ldots.)$

$(c_n) = (-1)^n/n$

How can we show (a_n) is increasing?

1. **Directly with algebra** Show $a_n \le a_{n+1}$ $\forall n$.

   OR $\dfrac{a_{n+1}}{a_n} \ge 1$ (if $a_n > 0$)

   OR $a_{n+1} - a_n \ge 0$

   Ex $\dfrac{n-1}{n}$ is incr'g:

   $a_{n+1} - a_n$ is

   $$\frac{(n+1)-1}{n+1} - \frac{n-1}{n} = \frac{n}{n+1} - \frac{n-1}{n} = \frac{n^2 - n^2 + 1}{n(n+1)} = \frac{1}{n(n+1)} \ge 0$$

2. **By induction**. Show $a_2 \ge a_1$. Assume $a_k \ge a_{k-1}$, show that $a_{k+1} \ge a_k$. (Example to come)

3. **Calculus**. If $a_n = f(n)$ "nice" formula $f(n)$, and $f' \ge 0 \Rightarrow f$ incr'g $\Rightarrow a_n$ incr'g.

Monotone Convergence Thm (4.3.3)

A monotone sequence is convergent iff it's bounded.

Pf Let s_n be monotone.

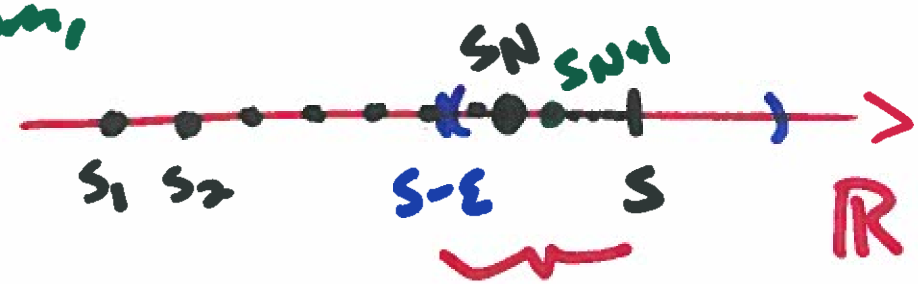
If s_n converges, we already know it's bdd.

(True for all convergent seq's - thm @ end of prev sect's)

In other direction, suppose s_n is bdd and show it converges. We'll do the case where s_n is increasing and bdd (above and below)

Suppose s_n incr'g and bounded above.

By completeness axiom, seq has a least upper bound. Call



$$s = \sup\{s_n\}$$

I claim $s_n \to s$. Let $\varepsilon > 0$. Need to show $\exists$ $N \ni n > N \Rightarrow |s_n - s| < \varepsilon$.

Since $s = \sup\{s_n\}$, $s - \varepsilon$ is <u>NOT</u> an upper bound

$\Rightarrow \exists$ s_N which is bigger than $s-\varepsilon$: $s - \varepsilon < s_N$.

s_n incr'g $\Rightarrow$ for all $n > N$, $s_n \geq s_N > s - \varepsilon$

But s upper bd to $\{s_n\}$ $\quad s - \varepsilon < s_n < s$

i.e. $|s_n - s| < \varepsilon$ $\forall$ $n > N$ so $s_n \to s$.