Thms in §§4.1–4.3 all apply to series (by applying them to the sequence of partial sums.

<u>Thm 8.1.6</u> $\sum a_n$ converges $\Leftrightarrow$ $s_n = \sum_{k=0}^{n} a_k$ converges

$\Updownarrow$ s_n Cauchy $\Leftrightarrow$

<u>Thm 8.1.4</u> If $\sum a_n = s$, $\sum b_n = t$, then

(a) $\sum (a_n + b_n) = \sum a_n + \sum b_n = s + t$

(b) $\sum (k a_n) = k \sum a_n = k \cdot s \quad \forall\, k \in \mathbb{R}$.

<u>Pf of (a)</u> [Sketch]

$\sum a_n = s \Leftrightarrow$ partial sums $s_n \to s$

$\sum b_n = t \Leftrightarrow$ partial sums $t_n \to t$.

$s_n + t_n =$ seq of partial sums of $\sum (a_n + b_n)$.

By Thm 4.2.1 (a), $\underline{s_n + t_n} \to s + t \Rightarrow \sum (a_n + b_n) = s + t$

⚠ Converse ($\Leftarrow$) of last thm not true:

$a_n = 1$, $\sum a_n = 1+1+1+1+\cdots = +\infty$

$b_n = -1$, $\sum b_n = -1-1-1-1-\cdots = -\infty$

$c_n = a_n + b_n = 0$, $\sum c_n = 0+0+0+\cdots = 0$

DO NOT split up a series $\sum (a_n + b_n)$

into $\sum a_n + \sum b_n$

unless you know the separate pieces converge.

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots$ ("The Harmonic Series.")

$s_1 = 1$

$s_2 = 3/2$

$s_3 = 11/6$

$s_4 = \cdots$

s_n increasing, but formula for s_n hard to find!

Does $\sum \frac{1}{n}$ converge? ($\Leftrightarrow$ does s_n converge?)

We know $s_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ converges $\iff$ s_n Cauchy

$\forall \varepsilon > 0 \ \exists N$ such that $n, m > N \Rightarrow |s_n - s_m| < \varepsilon$

We'll show s_n NOT Cauchy $\iff$ s_n diverges $\iff$ $\sum \frac{1}{n}$ too!

Suppose $m > n$, so $\frac{1}{m} < \frac{1}{n}$, $\frac{1}{m} < \frac{1}{n+1}, \ldots, \frac{1}{m} < \frac{1}{m-1}$

$$s_m - s_n = \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{m}\right) - \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)$$

$$= \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{m}$$

$$> \underbrace{\frac{1}{m} + \frac{1}{m} + \cdots + \frac{1}{m}}_{(m-n)}$$

$\Rightarrow s_m - s_n > \frac{(m-n)}{m} = 1 - \frac{n}{m}$. So if $m = 2n$, $s_m - s_n > 1 - \frac{1}{2}$
$= 1/2$.

If $\varepsilon = \frac{1}{2}$, there is no N s.t. $n, m > N \Rightarrow |s_m - s_n| < \varepsilon$. Not Cauchy

In other words, given <u>any</u> $M \in \mathbb{R}$, $\exists$ N such that:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{N} > M.$$

<u>But</u> $s_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ increases so slowly that these #'s are <u>enormous</u>.

n	s_n	$\gamma + \ln n$
1	1	0.577...
2	1.5	1.27
10	2.93	2.88
100	5.187...	5.182...
1000	7.485...	7.48497....

($\leftarrow$ where $1 + \frac{1}{2} + \cdots + \frac{1}{n} \approx \gamma + \ln n$, $\gamma \approx 0.577..$

So to get $s_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} > M$,

solve $\ln n + \gamma > M$

$n > e^{M - \gamma}$

$M = 100 \Rightarrow n > 10^{43}$

$M = 1000 \Rightarrow n > 10^{434}$

$M = 1\,000\,000 \Rightarrow n > 10^{484\text{[illegible]}}$

__Extra fun__ - does anything diverge __more__ slowly? Yes!

$$\sum_{p \text{ prime}} \frac{1}{p} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \cdots = +\infty$$

And yet,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}.$$

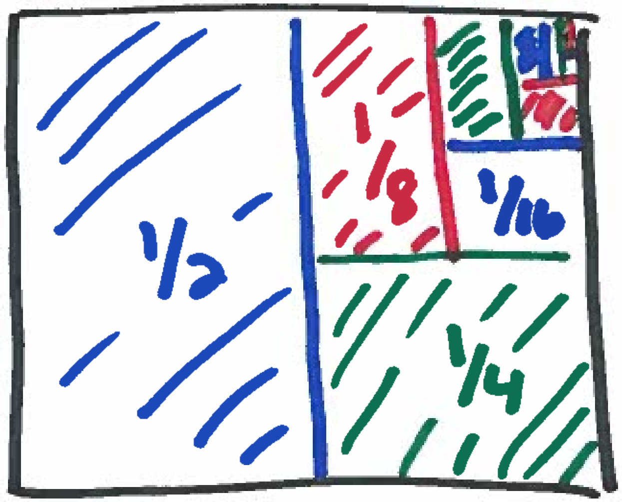
$$\sum_{n=1}^{\infty} \frac{1}{n^4} = 1 + \frac{1}{16} + \frac{1}{81} + \frac{1}{256} + \cdots = \frac{\pi^4}{90} \quad \leftarrow ?$$

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = 1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \cdots = \zeta(3) \approx 1.20205\ldots$$

(irrational; Apery)

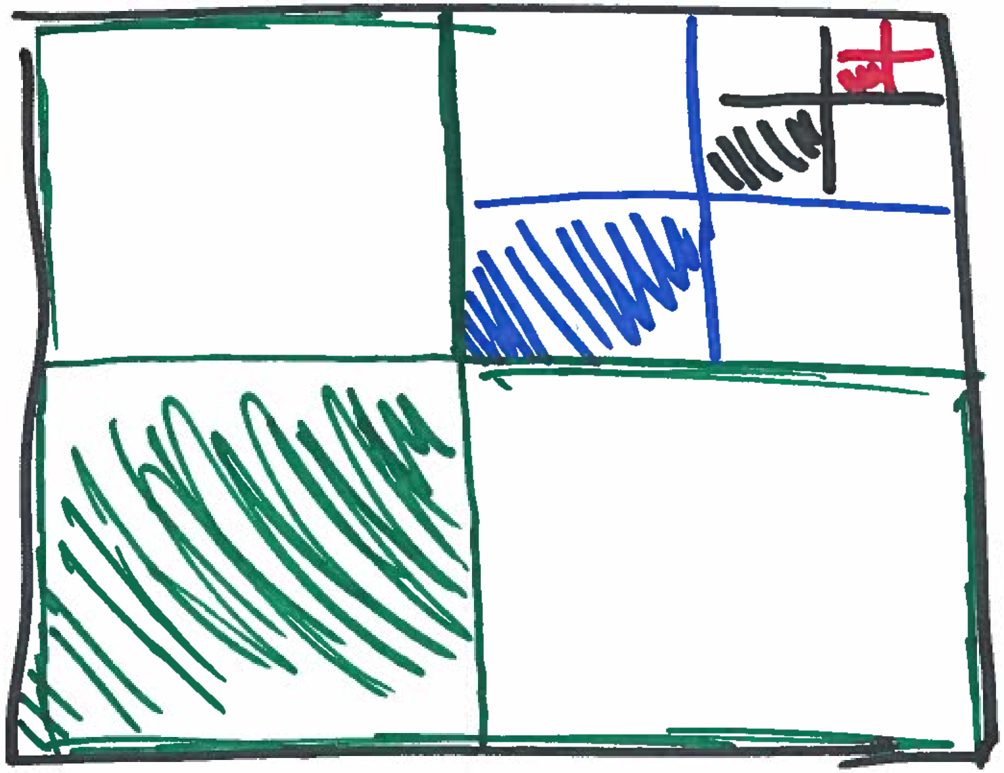
Geometric Repr'n of Series

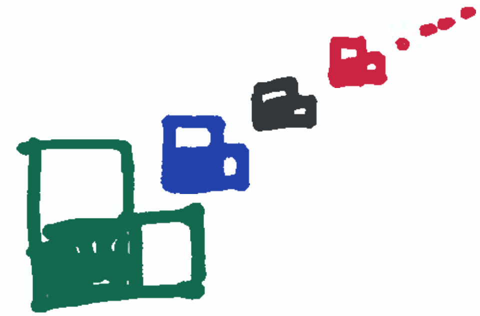
unit square:



$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1.$$

i.e. $$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 1$$



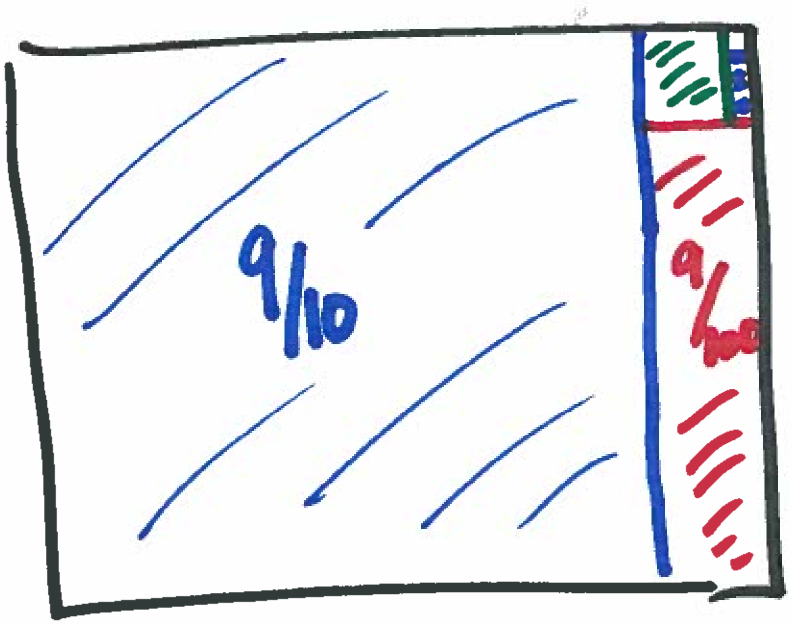


$\frac{1}{3}$ of L at each step

$$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \cdots = \frac{1}{3}$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n = \left(\sum_{n=0}^{\infty} \frac{1}{4^n}\right) - 1 = \frac{1}{1-1/4} - 1 = \frac{4}{3} - 1 = \frac{1}{3}.$$

Or....



$$\frac{9}{10}+\frac{9}{100}+\frac{9}{1000}+\cdots = 0.9+0.09+0.009+\cdots$$

$$= 0.99999\ldots$$

$$= 1.$$

Alt. Harmonic Series

(Proof that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2$$

uses a copyrighted image; I can't post the slide but I'll post a link on the webpage.)

Ex Telescoping Series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{4}} - \cdots\cdots = 1.$$

Note: $\dfrac{1}{n(n+1)} = \dfrac{1}{n} + \dfrac{(-1)}{n+1} \quad \left(= \dfrac{n+1-n}{n(n+1)} \right)$

$$S_1 = \left(\frac{1}{1} - \frac{1}{2}\right)$$

$$S_2 = \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \frac{1}{3}\right)$$

$$S_3 = \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{4}\right)$$

$$\vdots$$

$$S_n = \left(1 - \cancel{1/2}\right) \cancel{+} \cdots \cancel{+} \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}$$

$$\lim 1 - \frac{1}{n+1} = 1 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

<u>Wrapup</u>

<u>Thm</u> $\sum a_n$ converges $\Longrightarrow$ $a_n \to 0$

⚠ <u>Converse</u> ($a_n \to 0 \Rightarrow \sum a_n$ converges) is

<u>NOT TRUE!!</u>

<u>Ex</u> $\sum \frac{1}{n} = +\infty$ but $\frac{1}{n} \to 0$.

Intuitively, suppose $a_n \to L$. Eventually

$$\sum a_n \cong a_1 + \cdots + L + L + L + L + L \to \pm\infty$$