§8.2 Convergence Tests

Goal: Can we analyze a_n's instead of $S_n = a_1 + a_2 + a_3 + \cdots + a_n$ to determine if $\sum a_n$ converges?

Theorem names in this section might sound familiar.... comparison test, ratio test, root test, integral test, alternating series test.

Basic Test for divergence comes from

Thm $\sum a_n$ converges $\Rightarrow$ $a_n \to 0$

C.P. $a_n \not\to 0 \Rightarrow \sum a_n$ diverges.

Ex $$\sum_{n=1}^{\infty} \frac{(n+1)!}{(n-1)!} = \frac{2!}{0!} + \frac{3!}{1!} + \frac{4!}{2!} + \frac{5!}{3!} + \cdots\cdots$$

$$a_n = \frac{(n+1)!}{(n-1)!} = \frac{(n+1)\,n\,\cancel{(n-1)(n-2)\cdots\cdots 3\cdot 2\cdot 1}}{\cancel{(n-1)(n-2)\cdots\cdots 3\cdot 2\cdot 1}}$$

$$= n(n+1) = n^2+n \not\to 0.$$

⚠ Always worth checking b/c it's quick BUT most series $\sum a_n$ we'll see will have $a_n \to 0$.

Thm 8.2.1 (Comparison Test). $0 \le a_n \le b_n \ \forall n$.

(a) $\sum b_n$ converges $\Rightarrow$ $\sum a_n$ converges.

(b) $\sum a_n = +\infty$ $\Rightarrow$ $\sum b_n = +\infty$

⚠ book switches roles of a_n, b_n in (a)

Pf Let $s_n = a_1 + a_2 + \cdots + a_n$, $t_n = b_1 + b_2 + \cdots + b_n$ } $s_n \le t_n$ since $a_k \le b_k \ \forall k$.

(a) $a_n \ge 0 \Rightarrow s_n$ increasing.

By earlier sections, $s_n \le t_n \ \forall n \Rightarrow \lim s_n \le \lim t_n = t$

$t_n \to t$.

$\Rightarrow s_n$ bdd above by t, below by s_1 (since it's incr'g) $\Rightarrow$ by MCT s_n converges $\Rightarrow$ $\sum a_n$ converges

Key to using Comparison Test

(a) Compare to a known series.

(b) Make sure direction of comparison is useful.

<u>Ex</u> $0 \leq \frac{1}{n^2} \leq \frac{1}{n}$ $\forall n$ so

$$0 \leq \sum \frac{1}{n^2} \leq \sum \frac{1}{n} = +\infty$$

so comp. test tells us nothing about $\sum \frac{1}{n^2}$ in this case.

<u>Ex</u> $\sum \frac{1}{n(n+1)} = 1$ (telescoping)

$$\forall n \quad 0 \leq \frac{1}{n(n+1)} \leq \frac{1}{n^2}$$

$$\Rightarrow \quad 0 \leq \sum \frac{1}{n(n+1)} \leq \sum \frac{1}{n^2}$$

$$1 \leq \sum \frac{1}{n^2}$$ not a useful comparison.

$\exists$ useful comparisons!

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$ $\qquad 0 \le \dfrac{1}{(n+1)(n+1)} \le \dfrac{1}{n(n+1)} \quad \forall n$

and $\displaystyle\sum \frac{1}{n(n+1)}$ converges to 1

$\Rightarrow \displaystyle\sum \frac{1}{(n+1)^2}$ converges by comp. thm. (test).

To what?

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)^2} = \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots$$

$$= \underline{\left(\frac{1}{1} + \frac{1}{4} + \frac{1}{16} + \cdots\right)} - 1$$

$$= \frac{\pi^2}{6} - 1 \quad \text{(no proof)}$$

Alt.

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)^2} = \sum_{u=2}^{\infty} \frac{1}{u^2}$$

$u = n+1$

etc...

Thm If $\sum |a_n|$ converges, so does $\sum a_n$

Notes ① If $\sum |a_n|$ converges, we say the series $\sum a_n$ converges absolutely

② If $\sum a_n$ converges but $\sum |a_n|$ doesn't, we say $\sum a_n$ converges conditionally.

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2$ but $\displaystyle\sum \left|\frac{(-1)^{n+1}}{n}\right| = \sum \frac{1}{n} = +\infty$

Alt. Harmonic series converges conditionally, but NOT absolutely.

Pf that $\sum |a_n|$ converges $\Rightarrow$ $\sum a_n$ converges

Sketch of Pf: $s_n = a_1 + a_2 + \cdots + a_n$

$t_n = |a_1| + |a_2| + \cdots + |a_n|$

$a_n \le |a_n|$ and $s_n \le t_n$ $\forall n$, but comparison tests are useless.

$\sum |a_n|$ converges $\Leftrightarrow$ t_n converges $\Leftrightarrow$ t_n Cauchy. So we can make $t_n - t_m$ small.

$\boxed{\cancel{|x-y| \le \big||x|-|y|\big|}}$

Given $\varepsilon > 0$, $\exists$ N s.t. $n > m > N \Rightarrow$

$$|t_n - t_m| = \Big|\, |a_{m+1}| + |a_{m+2}| + \cdots + |a_n| \,\Big| < \varepsilon.$$

$$|s_n - s_m| = |a_{m+1} + a_{m+2} + \cdots + a_n| \le \Big| \uparrow \Big| < \varepsilon.$$

Thus $|s_n - s_m| < \varepsilon$.

Other Tests

Thm (Alternating Series Test) If $a_n \to 0$ and is decreasing then $\sum (-1)^{n+1} a_n$ converges.

[Ex: $\frac{1}{n}$ decr's, $\frac{1}{n} \to 0 \Rightarrow$ Alt. Harm. Series conv's]

Thm Ratio Test

(a) if $\lim \left|\frac{a_{n+1}}{a_n}\right| < 1$ then $\sum a_n$ converges absolutely.

(b) if $\lim \left|\frac{a_{n+1}}{a_n}\right| > 1$ then $\sum a_n$ diverges.

(c) otherwise, no information.

Thm Root Test

(a) if $\lim |a_n|^{1/n} < 1$ then $\sum a_n$ conv's abs.

(b) if $\lim |a_n|^{1/n} > 1$ then $\sum a_n$ diverges

(c) else no info!

$$\lim \left|\frac{a_{n+1}}{a_n}\right| = \lim |a_n|^{1/n}$$

Remember: The Ratio and Root Tests in Section 8.2 use something called “lim inf” and “lim sup.”

We didn’t cover Section 4.4 on Subsequences, which is where “lim inf” and “lim sup” are introduced.

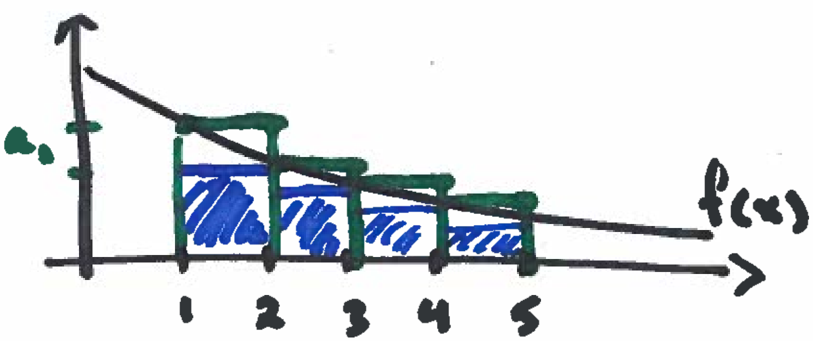
Thus I gave you variants of the Ratio and Root Tests which use regular limits; those variants are what we will use on the Final Exam.

Integral Test Let $a_n = f(n)$, where $f: [0,\infty) \to \mathbb{R}$ is positive, continuous and decreasing. Then

$$\sum a_n = \sum f(n) \text{ converges} \iff \int_1^\infty f(x)\,dx \text{ converges}$$

Recall $\displaystyle\int_1^\infty f(x)\,dx = \lim_{b\to\infty} \int_1^b f(x)\,dx$

(Sketch of) Pf



Using left endpts,

$$a_1 + a_2 + a_3 + \cdots + a_n \geq \int_1^n f(x)\,dx$$

if $\int_1^\infty f(x)\,dx = +\infty$, $\sum a_n = +\infty$ too.

Using right endpts

$$a_2 + a_3 + \cdots + a_n \leq \int_1^n f(x)\,dx.$$

if $\int_1^\infty f(x)\,dx$ is finite $\sum a_n$ converges too.

$$\lim\left|\frac{a_{n+1}}{a_n}\right|,\quad \lim|a_n|^{1/n},\quad \lim\left(\int_1^n f(x)\,dx\right)$$

$\sum_{n=1}^{\infty}\left(\frac{3}{n}\right)^n$ Ratio $\left|\frac{(3/n+1)^{n+1}}{(3/n)^n}\right| = \frac{3^{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{3^n} = \frac{3n^n}{(n+1)^{n+1}}$

Root $\left|\left(\frac{3}{n}\right)^n\right|^{1/n} = \frac{3}{n} \to 0 < 1 \Rightarrow \sum\left(\frac{3}{n}\right)^n$ conv's absolutely

$\sum_{n=1}^{\infty}\frac{3^n}{n!}$ Ratio $\frac{3^{n+1}}{(n+1)!}\cdot\frac{n!}{3^n} = \frac{3}{n+1} \to 0 < 1 \Rightarrow$ conv. absolutely.

$\sum_{n=1}^{\infty}\frac{1}{n^3}$ $\int_1^n \frac{1}{x^3}dx = \left.\frac{-x^{-2}}{2}\right|_1^n = -\frac{1}{2n^2}+\frac{1}{2} \to \frac{1}{2}$ as $n\to+\infty$.

$\Rightarrow \sum\frac{1}{n^3}$ converges too!

$\sum_{n=1}^{\infty}\frac{n-1}{3n+2}$ $\frac{n-1}{3n+2} \to \frac{1}{3} \neq 0 \Rightarrow \sum\frac{n-1}{3n+2}$ div's.

$\sum_{n=1}^{\infty}(-1)^n\frac{n^2+2}{3n^3}$ $\frac{n^2+2}{3n^3} = \frac{\frac{1}{n}+\frac{2}{n^3}}{3}$ Check with. A.S.T.

<u>Ex</u> More generally, $\sum \frac{1}{n^p}$ is called a p-<u>series</u>.

Using integral test:

$$\int_1^n \frac{1}{x^p}\,dx = \int_1^n x^{-p}\,dx$$

$$= \frac{x^{-p+1}}{1-p}\Big|_1^n = \frac{n^{1-p}}{1-p} - \frac{1}{1-p}$$

∴ this converges as $n \to \infty$ depending on n^{1-p} term.

As $n \to \infty$ this quantity is <u>finite</u> if $1-p<0$ $(p>1)$

<u>infinite</u> if $1-p>0$ $(p<1)$

By int test $\sum \frac{1}{n^p}$ converges if $p>1$, div's if $p<1$

($p=1$: Harmonic Series, diverges)

Given $\sum a_n$, which test to use?

- If $a_n \not\to 0$ then $\sum a_n$ diverges.
- Known form: geometric series, p-series — use formulas.

[• Check if it's telescoping series]

- If a_n has $(-1)^n$ or $(-1)^{n+1}$ (etc), Alt. Series Test.
- If it's "similar" to known series (geo, p, harm) try comparison test:

$$\underline{\text{Ex}} \quad \sum \frac{1}{3+2^n} \leq \sum \frac{1}{2^n}$$

- If n appears as exp (and especially in base too), root test:

$$\sum \left(\frac{1}{n}\right)^n \text{ etc.}$$

- Factorials, exp formulas: Ratio Test.