§1.2 Quantifiers

Quantifiers... quantify things!

In math, we're usually interested in whether something is true:

- always $\qquad x^2 \geq 0$
- sometimes $\qquad x^2 > 0$
  (at least once)
- never $\qquad x^2 < 0$

efficient
Since we're ~~lazy~~, we use symbols...

Existential Quantifiers

$\exists$: there exists (at least one)
$\exists!$: there exists a unique (exactly one) (not universal)
$\nexists$: there does not exist (slang)

Universal Quantifier

$\forall$: for all, for every.

Other Notation

$\ni$ $p(x)$

$\ni$: such that (sometimes $:$, $|$, esp. with sets)
$p(x)$: stmt whose truth value depends on value of x.

Ex $p(x): x^2-1=0$ $\quad$ $p(1)$: true
$p(2)$: false

Ex Write these stmts with symbols

Ⓣ For some x, $x^2-1=0$. $\exists x\ (\in\mathbb{R}) \ni x^2-1$

Ⓣ For every real number $x>0$, there is a real number y such that $y^2=x$.

$\forall x>0, \exists y \ni y^2=x$. OR $\forall x>0, \exists y\in\mathbb{R} \ni y^2=x$ OR

Ⓣ Every real number has a cube root.

$\forall x, \exists y \ni y^3=x$. $\forall x\ \exists x^{1/3}$

Ⓣ Given any #, there is a larger #.

$\forall x\ \exists y \ni y>x$ (y can depend on x)

There exists a largest #. F

$\exists y \ni \forall x,\ y > x$

order of quantifiers is important!

There is no square root of -2 in $\mathbb{R}$. T

slang: $\nexists x \in \mathbb{R} \overset{\text{s.t.}}{\ni} x^2 = -2$

Better: $\forall x \in \mathbb{R},\ x^2 \neq -2$.

A word about variables

x, y are assumed to be real unless otherwise specified in this course.

⚠ Negation of quantifiers is tricky!

In words, (assuming ~rainy = sunny)

Negation of "Every day is rainy" isn't "Every day is sunny."

It's: "At least one day is sunny".

Symbolically, negation of $\forall x,\ p(x)$ is $\exists x \ni \sim p(x)$

i.e.

$$\boxed{\begin{aligned} &\sim[\forall x,\ p(x)] \Leftrightarrow \exists x \ni \sim p(x) \\ &\sim[\exists x \ni p(x)] \Leftrightarrow \forall x, \sim p(x) \end{aligned}}$$

<u>Ex</u> Negate:

(a) $\forall x,\ g(x) > 0$

$\exists x \ni \sim[g(x) > 0]$
$\exists x \ni g(x) \leq 0$

(b) $\exists x \ni f'(x) = 0$

$\forall x,\ f'(x) \neq 0.$

(c) $\forall x, (\exists y \ni y > x).$

$\exists x \ni \sim[\exists y \ni y > x]$
$\exists x \ni \forall y,\ y \leq x.$

⚠ Don't go overboard!

Negate:

there's an implicit "$\forall x$" here

$\forall \varepsilon > 0 \; \exists \delta \ni 0 < |x-a| < \delta \Rightarrow |f(x) - f(a)| < L.$

means: $\lim_{x \to a} f(x) = L$ (later this semester)

Negation is $\lim_{x \to a} f(x) \neq L$

$\exists \varepsilon > 0 \ni \sim[\,.\; - - - -\,]$

$\exists \varepsilon > 0 \ni \forall \delta, \sim[0 < |x-a| < \delta \Rightarrow |f(x) - f(a)| < L]$

$\exists \varepsilon > 0 \ni \forall \delta, \exists x \ni 0 < |x-a| < \delta$ and $|f(x) - f(a)| \geq L$