§2.1 Basic Set Theory

Set theory can seem tedious at first, but it's <u>essential</u>.

In higher level math courses, set thy replaces arithmetic/algebra as language of math.

Read this section carefully, even (or especially?) if you know some set theory.

<u>Def</u> A <u>set</u> is an <u>unordered</u> collection of objects, called <u>elements</u> (elts). Write $x \in A$ to denote x is a member of A. If A has <u>finite</u> # of elts, $|A|$ = # of elts in A = cardinality of A.

Ways to Define Sets

By listing elts:

$$A = \{1, 2, 3, 4, 5\}$$
$$B = \{\bigcirc, \triangle, \square\}$$

Via Defining Property:

$$C = \{x \mid x > 0\} = \{x : x > 0\} \left(= \{x > 0\}\right)$$

"slang"

Note: A "universal set" is often implied or assumed.

A: integers? B: shapes? C: real #'s?

Standard Names

Standard Names

$\mathbb{N} = \{1,2,3,4,5,\dots\} = \{1,2,3,\dots\}$ not $\{1,\dots\}$

⚠ $0 \notin \mathbb{N}$ as defined ↺ but sometimes is! (Grr....)

Sometimes: $\mathbb{N}_0 = \{0,1,2,3,\dots\}$

$\mathbb{Z} = \{\dots,-2,-1,0,1,2,\dots\}$ = integers. (German: Zahlen)

$\mathbb{Q}$ = rat'l #'s

$\mathbb{R}$ = real #'s

$\mathbb{C}$ = complex #'s
$= \{a+bi \mid a,b \in \mathbb{R},\ i^2=-1\}$

"Blackboard Bold"

Also, $\mathbb{H}$, $\mathbb{F}_{p^n}$

$\mathcal{C}^1$ = fns with cont. deriv.

$\emptyset = \{\ \} =$ empty set

$(1,3] = \{x : 1 < x \leq 3\}$

Def A is a subset of B, $A \subseteq B$, if $x \in A \Rightarrow x \in B$

Ex $B = \{1,2,3,4\}$

$\{1,3,4,2\} \subseteq B$ non-proper

$\{1,3\} \subseteq B$ proper

$\{1,2,5,6\}$ no! $(5 \notin B,\ 6 \notin B)$

$\{1,1,1,1\} = \{1\} \subseteq B$

$\emptyset \subseteq B$

$\underline{x \in \emptyset} \Rightarrow x \in \{1,2,3,4\}$

F so impl'n is true, def^n satisf'd.

Def A subset of B is proper if it doesn't contain all elts of B, i.e. $A \subseteq B$ and $B \nsubseteq A$.

Notes ① to show $A = B$, prove $A \subseteq B$ and $B \subseteq A$

Think: $x = y$ iff $x \leq y$ and $y \leq x$

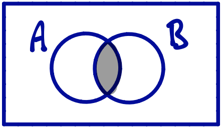
② Some books use $\subset$, $\subseteq$ for "proper" and "proper or equal", like $<$, $\leq$. Most use $\subset$ for both!! Ours uses $\subseteq$.

Forming New Sets from Old

Venn Diagram

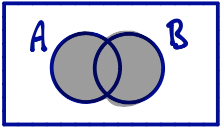
Intersection

$$A \cap B = \{x \mid x \in A \wedge x \in B\} = B \cap A$$



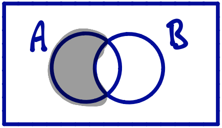
Union

$$A \cup B = \{x \mid x \in A \vee x \in B\} = B \cup A$$



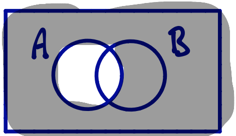
Set Difference

$$A - B = A \setminus B = \{x \mid x \in A \wedge x \notin B\} \neq B \setminus A$$



Complement - with universal set X

$$A^c = \overline{A} = \{x \mid x \notin A\} = X \setminus A$$



<u>Ex</u> In $\mathbb{N}$, let $A =$ even natural #'s, $B = \{1,2,3,4,5,\ldots,10\}$

$A \cap B = \{2,4,6,8,10\}$

$A \cup B = A \cup \{1,2,3,\ldots,10\} = A \cup \{1,3,5,7,9\} = \{1,2,\ldots,10\} \cup \{12,14,16,\ldots\}$

$\overline{A} = A^c =$ odd #'s $= \{1,3,5,\ldots\}$

$A \setminus B = \{12,14,16,\ldots\}$

$B \setminus A = \{1,3,5,7,9\}$

$A \cup \emptyset = A$

$B \cap \emptyset = \emptyset$

Ex Prove: $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$ (Distributive)

Think: $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$

Pf We must show $X \cap (Y \cup Z) \subset (X \cap Y) \cup (X \cap Z)$

and $(X \cap Y) \cup (X \cap Z) \subset X \cap (Y \cup Z)$

Let's do the 1st inclusion, LHS $\subset$ RHS

Let $x \in X \cap (Y \cup Z)$. Thus $x \in X$ (*) and $x \in (Y \cup Z)$, which means it's in Y or it's in Z.

If $x \in Y$, then $x \in X \cap Y$, since $x \in X$ as well. (*) Similarly, if $x \in Z$, then $x \in X \cap Z$.

Thus $x \in X \cap Y$ or $x \in X \cap Z$, which means $x \in (X \cap Y) \cup (X \cap Z)$.

Hence $X \cap (Y \cup Z) \subset (X \cap Y) \cup (X \cap Z)$.

Now we show $(X \cap Y) \cup (X \cap Z) \subset X \cap (Y \cup Z)$. (sketch of proof)

Let $x \in (X \cap Y) \cup (X \cap Z)$. $\Rightarrow x \in (X \cap Y)$ or $x \in (X \cap Z)$

<u>Case 1</u>: $x \in X \cap Y \Rightarrow x \in X$ and $x \in Y$
<u>Case 2</u>: $x \in X \cap Z \Rightarrow x \in X$ and $x \in Z$.

Because $x \in X$ in both cases, and one case must be true, <u>$x \in X$</u>
Either $x \in Y$ or $x \in Z$ (depending on which case) [or both....]
So $x \in Y \cup Z$.

Hence $x \in X$ and $x \in Y \cup Z$.

.... Thus $x \in X \cap (Y \cup Z)$

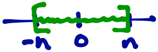
Having shown both inclusions, we can say

$$(X \cap Y) \cup (X \cap Z) = X \cap (Y \cup Z)$$

Indexed Sets Often we use families of sets

Ex $A_n = [-n, n]$, $n \in \mathbb{N}$: n is "index", $\mathbb{N}$ indexing set

$A_1 = [-1, 1]$
$A_2 = [-2, 2]$ $A_{100} = [-100, 100]$



We often use notation similar to

$$\sum_{n=1}^{r} a_n = a_1 + a_2 + \cdots + a_r$$

with indexed sets.

$$\bigcup_{n=1}^{5} A_n = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 = A_1 \cup A_2 \cup \cdots \cup A_5$$

$$\bigcap_{n=1}^{\infty} A_n = A_1 \cap A_2 \cap A_3 \cap \cdots$$

Today's Math Fun Fact:

The Ham Sandwich Theorem.

Take two pieces of bread and a slab of ham and place them <u>anywhere in the universe</u>!

With one slice of a (very long...) knife you can simultaneously cut all three in half!