§2.2 Relations

Or: "Sophisticated definitions of things you (mostly) already know."

$\exists$ five definitions in this section.

1. Ordered Pair
2. Cartesian Product
3. Relation
4. Equivalence Relation
5. (partition into) Equiv. Classes

Sets are Unordered - but often order matters!

With pts/vectors: $(1,2) \neq (2,1)$. $\langle 3,4,5 \rangle \neq \langle 5,4,3 \rangle$

Option 1 Define a new "ordered set".

Option 2 Mathematicians like building everything out of a few basic objects.

Def The ordered pair (a,b) is the set

$$(a,b) = \{\{a\},\{a,b\}\} = \{\{a,b\},\{a\}\}$$
$$= \{\{b,a\},\{a\}\}$$

Ex $(1,2) = \{\{1\},\{1,2\}\}$
$\quad\quad\quad\quad\quad\quad\quad\neq \quad\quad =$ $\quad\not\Rightarrow$
$(2,1) = \{\{2\},\{2,1\}\}$

Thm $(a,b)=(c,d) \iff a=c$ and $b=d$

Pf $\Leftarrow$ Suppose $a=c$ and $b=d$. Then

$$\begin{aligned} (a,b) &= \{\{a\},\{a,b\}\} && \text{by def}^{\underline{n}} \text{ of } (a,b) \\ &= \{\{c\},\{c,d\}\} && \text{by assumption} \\ &= (c,d) && \text{by def}^{\underline{n}} \end{aligned}$$

$\Rightarrow$ (You try - book)

Def The **Cartesian Product** of sets A, B is

$$A \times B = \{ (a,b) \mid a \in A,\ b \in B \}$$

Examples points and vectors live in

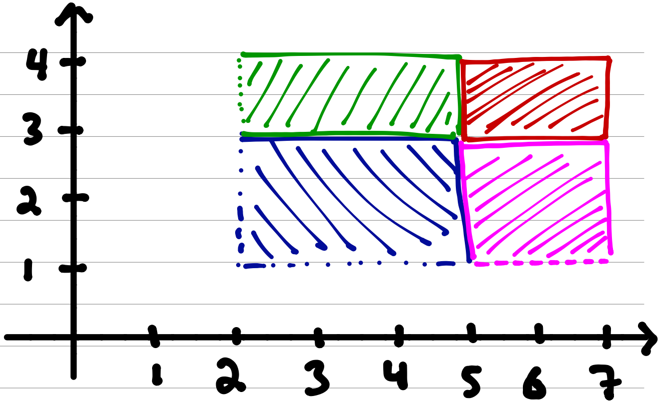
$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{ (x,y) \mid x \in \mathbb{R} \text{ and } y \in \mathbb{R} \}$$

$$\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R} = \underline{(\mathbb{R} \times \mathbb{R})} \times \underline{\mathbb{R}} = \{ (\underline{(x,y)}, \underline{z}) \mid (x,y) \in \mathbb{R}^2, z \in \mathbb{R} \}$$

$$\text{“}=\text{”} \{ (x,y,z) \mid x,y,z \in \mathbb{R} \}$$

Ex

$A = (2,5]$
$B = (1,3]$
$C = [5,7]$
$D = [3,4]$



$A \times B = (2,5] \times (1,3] = \{(x,y) \mid 2 < x \leq 5 \text{ and } 1 < y \leq 3\}$

$A \times D = \{(x,y) \mid x \in (2,5] \text{ and } y \in [3,4]$

$C \times B = \{(x,y) \mid x \in [5,7] \text{ and } y \in (1,3]\}$

$C \times D = \{(x,y) \mid 5 \leq x \leq 7 \text{ and } 3 \leq y \leq 4\}$

Relations

Often we're interested in relationships b/w elts of sets:

With #'s $a<b$, $x \geq y$, $p=q$

With Shapes $\triangle ABC \cong \triangle DEF$, or $\triangle ABC \sim \triangle DEF$
(or same area)

Technical Def

A relation between A and B is subset $R \subset A \times B$. If $(a,b) \in R$ we write aRb (often replace R w/ symbol) and say "a is related to b"

Notes ① If $A=B$, we say R is relation on A.

② $aRb \Rightarrow (a,b) \in R \subset A \times B \neq B \times A$

⚠ and b not nec. related to a!

Ex $P = \{A, B, C, D\}$

$S = \{$house, car, cat, dog, bike, grapes$\}$

A owns house
B owns house $\Big\}$ A,B married?
C owns cat, dog, grapes

"owns" is a relation. As a subset of $P \times S$,

owns $= \{$(A, house), (B, house), (C, cat), (C, dog), (C, grapes)$\}$

Notice: D owns nothing, nobody owns car, bike.

$\underline{\text{Ex}}$ $=$ is a relation on $\mathbb{Z}$, given by

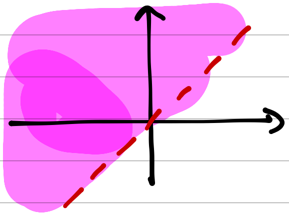
$$\{\ldots, (-2,-2), (-1,-1), (0,0), (1,1), (2,2) \ldots\} \subseteq \mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$$

$\underline{\text{Ex}}$ $<$ is a rel'n on $\mathbb{R}$, repr'd by

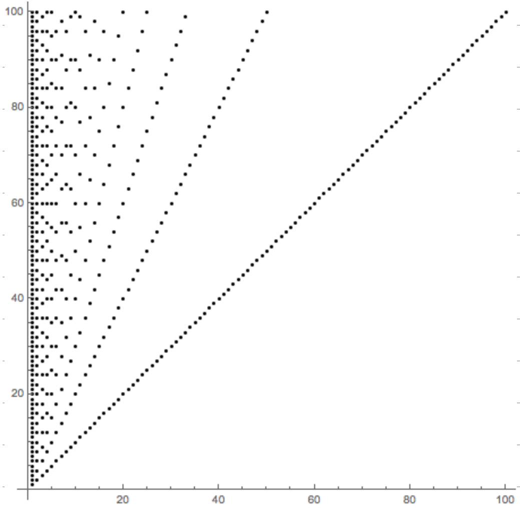
$$< = \{ (0,1), (0,2), (\cancel{0,} 0.9), (0.09), \ldots \}$$

$$= \{ (x,y) \in \mathbb{R}^2 \mid x < y \}$$

graphically



Ex What relation on $\mathbb{N}$ is represented in this picture?



aRb iff $a | b$, i.e a divides b with no remainder

Ex Clock arithmetic, relation on $\mathbb{Z}$.

informally: every time we hit 12, we wrap back around to 0.

$a \equiv b$ or $a = b \bmod 12$ iff:

(1) a and b have same remainder when divided by 12:

$10 \div 12 = 0 \ R\ 10$ $\qquad 10 \equiv 22 \equiv 46$

$22 \div 12 = 1 \ R\ 10$

$46 \div 12 = 3 \ R\ 10$

(2) OR $a = b + 12k$ for some $k \in \mathbb{Z}$

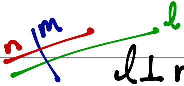
Def an equivalence relation R is on a set S which satisfies these three conditions:

$\forall x,y,z \in S$: (1) Reflexive: xRx
(2) Symmetric: $xRy \Rightarrow yRx$
(3) Transitive xRy and $yRz \Rightarrow xRz$

Ex Which of these are equiv. rel'ns?

- $\mathbb{R}$, $=$ YES! (motivating example!)
- $\mathbb{N}$, $<$ No! not reflexive. ($x \not< x$)
- $\mathbb{N}$, $\leq$ No! refl've, trans've, not symmetric. ($4 \leq 10$, $10 \not\leq 4$)
- $\mathbb{N}$, $=$ Yes!
- $\mathbb{N}$, $|$ (divides) No! refl've, trans've, not symmetric.

- Polygons, $\cong$ (congruent) Yes
- Polygons, $\sim$ (similar) Yes
- Lines, $\parallel$ (parallel) Yes
- Lines, $\perp$ (perp.) No! not reflive: ℓ not $\perp$ ℓ



$\ell \perp m$, $m \perp n$
ℓ **not** $\perp$ n

- $\mathbb{Z}$, $\equiv$ Yes! Let's prove the 3 conditions!

(1) Reflexive: $n \equiv n$ $\forall\, n \in \mathbb{Z}$.

Let $n \in \mathbb{Z}$. Then $n = n + 12(0)$, so $n \equiv n$.

(2) Symmetric.

Suppose $n \equiv m$. Then $n = m + 12k$, some $k \in \mathbb{Z}$.
Then $n - 12k = m$, or $m = n + 12(-k)$. Thus $m \equiv n$.

(3) Transitive.

Suppose $n \equiv m$, $m \equiv p$, so $n = m + 12k$ and $m = p + 12\ell$, $k, \ell \in \mathbb{Z}$.
Then $n = (p + 12\ell) + 12k = p + 12(k + \ell)$, so $n \equiv p$.

Fall '11 Exam! Consider this relation on $\mathbb{R}^2$:

$$(a,b)\ R\ (c,d) \text{ iff } a^2+b^2=c^2+d^2$$

(a) Prove R is an equiv. rel'n. We need to show R is:

Reflexive Let $(a,b)\in\mathbb{R}^2$. Then $a^2+b^2=a^2+b^2$, so $(a,b)\,R\,(a,b)$.

Symmetric Let (a,b) and (c,d) be in $\mathbb{R}^2$ such that $(a,b)\,R\,(c,d)$.
Then $a^2+b^2=\underline{c^2+d^2}$
and $\underline{c^2+d^2}=a^2+b^2$ so $(c,d)\,R\,(a,b)$.
(because equality of #'s is symmetric)

Transitive Let $(a,b)\,R\,(c,d)$ and $(c,d)\,R\,(e,f)$.
Then $a^2+b^2=c^2+d^2$
and $c^2+d^2=e^2+f^2$. Because equality of #'s is trans, $a^2+b^2=e^2+f^2$, and $(a,b)\,R\,(e,f)$.

More Abstract Example (like 2.2 #30)

Let $S=\{a,b,c,d\}$. What is the eq. rel'n R on S with the fewest members such that $\underbrace{(a,b)}_{aRb}, \underbrace{(c,b)}_{cRb} \in R$?

$R \subset S \times S = \{(x,y) : x,y \in S\}$

$R = \{ \underbrace{(a,a), (b,b), (c,c), (d,d)}_{R \text{ reflexive}}, \underbrace{(a,b), (c,b), (b,a), (b,c)}_{\text{given} \quad \text{symm.}}, \underbrace{(a,c)}_{\text{trans}}, \underbrace{(c,a)}_{\text{trans/symm}} \}$

⚠ We don't often do problems like this – but it's a good way to see how well you understand the technical definitions.

⚠ In this example we use set-theoretic notation for rel'n, $R \subset S \times S$, so elts of R were pairs: $(a,b) \in R \Leftrightarrow aRb$

In prev example, $(a,b) R (c,d) \Leftrightarrow ((a,b),(c,d)) \in R \subset \mathbb{R}^2 \times \mathbb{R}^2$

$\boxed{\text{Equivalence Rel'ns are our way of generalizing "equality" to other contexts, like geometry}}$

Given an equiv. rel'n, an important question is:

For a certain x, what is x related to??

<u>Def</u> Given an eq. rel'n R on a set S, the equivalence class of $x \in S$ is:

$$E_x = \text{everything related (equiv.) to } x$$

$$= \{y : y \sim x\}$$

$E_x = \{y \in S : xRy\}$

<u>Ex</u> Clock Arithmetic

$E_0 = \{\ldots, -12, 0, 12, 24, 36, 48, \ldots\}$

$E_1 = \{\ldots, -11, 1, 13, 25, 37, 49, \ldots\} = E_{-11} = E_{49}$ etc.

$E_2 = \{\ldots, -10, 2, 14, 26, 38, 50, \ldots\}$

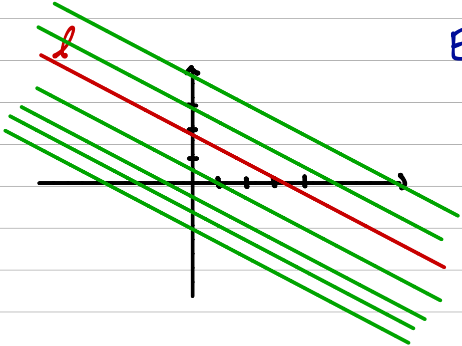
$\vdots$

$E_{11} = \{\ldots, -1, 11, 23, 35, \ldots\}$

$E_{12} = E_0$

$E_x = \{ y \in S : x R y \}$

<u>Ex</u> $2x + 3y = 6$ ($\leftarrow$ these x's, y's different than)



Equiv rel'n on lines in $\mathbb{R}^2$: $\parallel$

E_ℓ = any line $\parallel \ell$
= any line w/ slope $-\frac{2}{3}$

Fall '11 Exam! Consider this relation on $\mathbb{R}^2$:

$$(a,b)\ R\ (c,d) \text{ iff } a^2+b^2=c^2+d^2$$

(b) Describe the equivalence class of $(3,4)$. What does it look like geometrically?

$$E_{(3,4)} = \{(x,y) : (x,y) \sim (3,4)\}$$
$$= \{(x,y) : x^2+y^2 = 3^2+4^2\}$$
$$= \{(x,y) : x^2+y^2 = 5^2\}$$

all pts on circle of rad 5, ct'd @ origin.