Thm 2.3.20 Let $f: A \to B$ and $g: B \to C$

(a) f, g surjective $\Rightarrow$ $g \circ f$ surjective

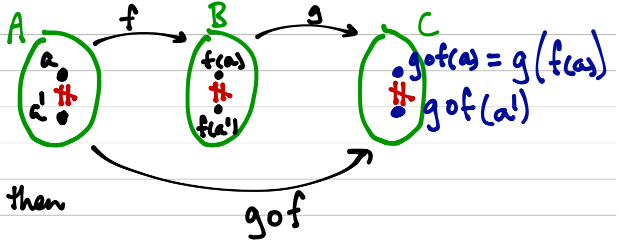
(b) f, g injective $\Rightarrow$ $g \circ f$ injective

(c) f, g bijective $\Rightarrow$ $g \circ f$ is bijective. by (a), (b) — on WQ - need to show (a,b).

Thm 2.3.20 Let $f: A \to B$ and $g: B \to C$

(b) f, g injective $\Rightarrow$ $g \circ f$ injective

Pf (Sketch!)



Def $g \circ f$ injective means: if $g \circ f(a) = g \circ f(a')$, then $a = a'$

Better: contrapositive: $a \neq a' \Rightarrow g \circ f(a) \neq g \circ f(a')$. (Want to show)

Let $a \neq a'$ in A. Then $f(a) \neq f(a')$, because f is injective.
Since g is also injective, $g(f(a)) \neq g(f(a'))$.

Functions Acting on Sets

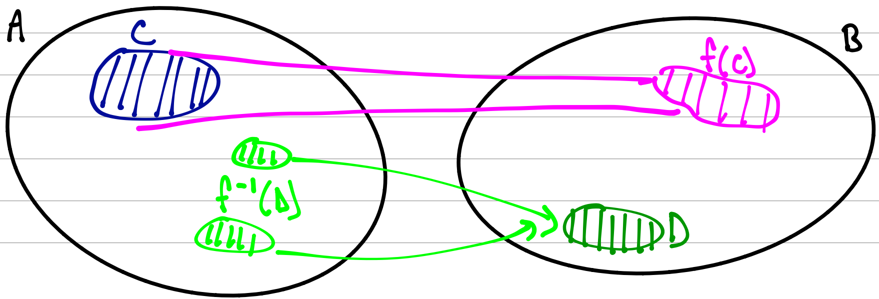
Def Let $f: A \to B$ and suppose $C \subseteq A$, $D \subseteq B$. Then

$$f(C) = \{b \in B : b = f(c) \text{ for some } c \in C\}$$ "image of C"

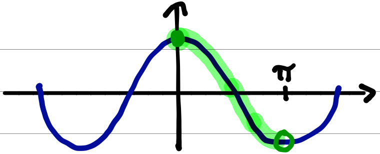
$$= \{b \in B : \exists c \in C \text{ s.t. } f(c) = b\}$$

$$= \{f(c) : c \in C\}$$

$$f^{-1}(D) = \{a : f(a) \in D\}$$ "pre-image of D"

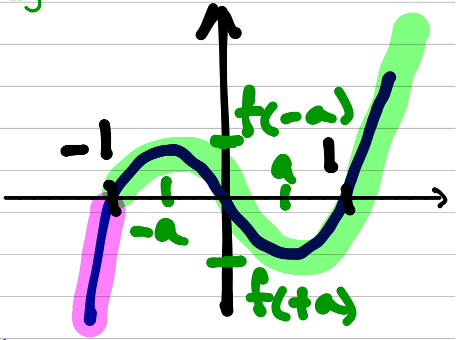


Ex $f(x) = \cos x$



$f([0,\pi)) = (-1, 1]$

$f^{-1}(\{0\}) = \{ \frac{\pi}{2} + \pi k : k \in \mathbb{Z} \}$



Ex $f(x) = x^3 - x = x(x-1)(x+1)$

$f((-\infty,-1]) = (-\infty, 0]$

$f((-1,\infty)) = [f(a), \infty) = f((0,\infty))$

$f((1,\infty)) = (0,\infty)$

$f^{-1}((0,\infty)) = (-1,0) \cup (1,\infty)$

$\exists$ many theorems involving all of the concepts:

Thm 2.3.16

(a) $C \subseteq f^{-1}[f(C)]$

(b) $f[f^{-1}(D)] \subseteq D$

(c) $f(C_1 \cap C_2) \subseteq f(C_1) \cap f(C_2)$

$\vdots$

Thm 2.3.18

$\vdots$

(c) if f injective $f(C_1 \cap C_2) = f(C_1) \cap f(C_2)$

⚠ Assignment: read through these thms and come to class on Tuesday with questions!

Last Concept in §2.3

Def Let $f: A \to B$ be bijective. The inverse of f is the function $f^{-1}: B \to A$ which "undoes" f:

$$f^{-1}(b) = a \text{ where } f(a) = b$$

Relation version: $(b,a) \in B \times A$ is in f^{-1} if $(a,b) \in f \subseteq A \times B$

$$f^{-1} = \{ (b,a) : (a,b) \in f \}.$$

⚠ Notice the similarity in notation with pre-images.

preimages generally written for sets: e.g. $f^{-1}(D)$

$f^{-1}(\{x\})$

Thm 2.3.24 Let $f: A \to B$ be bijective. Then

(a) $f^{-1}: B \to A$ is also a bijection.

(b) $f^{-1} \circ f = id_A$ and $f \circ f^{-1} = id_B$

where $id_S: S \longrightarrow S$ is identity function
$x \longmapsto x$

Ex $f: \mathbb{Z} \to \mathbb{Z}$
$n \mapsto n+1$

$f^{-1}: \mathbb{Z} \to \mathbb{Z}$
$m \mapsto m-1$