§2.3 Functions

This section contains <u>many</u> def$\underline{^s}$, including "function" as a relation, i.e. a subset of a Cartesian Product.

We'll cover this section in depth. Not all def$\underline{^s}$/concepts are vital, but you will make your future mathematical lives easier if you put in the effort to learn:

☆ 1. $f: A \to B$ notation, domain, codomain, range

☆☆ 2. injective, surjective, bijective

3. fn inverses, preimages

4. Compositions (and interplay with #1-3)

Algebra Through Calculus A function is a formula or rule which takes each input and transforms it to an output.

inputs: x, t, θ outputs $f(x), g(t), y=\sin(\theta)$

MV Calc/Linear Alg/etc. We often use a more general notation

$f: A \to B$

A: set of inputs, domain

B: set of (possible) outputs, codomain target space

range(f) = rng(f) = set of actual output $= \{f(a) \mid a \in A\}$
$= \{b \in B \mid \exists a \ni f(a) = b\}$

In many books, range = pot. outputs = codomain, and image is actual outputs.

f must assign exactly one value in B to each elt in A.

Ex $f: \mathbb{R}^2 \to \mathbb{R}$ $\qquad f(x,y) = x^2 - y^2$

$(x,y) \mapsto x^2 - y^2$

inputs are $(x,y) \in \mathbb{R}^2$

outputs are $x^2 - y^2 \in \mathbb{R}$.

Ex $f: \mathbb{Z} \to \mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, 3, \dots\}$

$m \mapsto m^2$

Range = actual outputs = $\{0, 1, 4, 9, \dots\}$

Could also write:

$f: \mathbb{Z} \to \mathbb{Z}$

$m \mapsto m^2$

$f: \mathbb{Z} \to \{0, 1, 4, 9, \dots\}$

$m \mapsto m^2$

Also: $f: \mathbb{N}_0 \to \mathbb{N}_0$

$m \mapsto m^2$

has same range.

⚠ This book is even more general - at least, at first...

Def A function from A to B is a non-empty (subset) relation $f \subseteq A \times B$ s.t.

(1) $\forall a \in A, \exists b \in B \ni (a,b) \in f$

(2) if (a,b) and (a,b') are in f, $b = b'$

Instead of giving a formula or rule, this method lists all inputs with their corresponding outputs:

$(a,b) \in f$ means "$f(a) = b$"

(1) says every a in domain gets assigned a fn value.

(2) says every a gets sent to only one $f(a)$

Ex $f: \mathbb{N} \to \mathbb{N}$ e.g. $f(n) = n+1$

$n \mapsto n+1$

As subset of $\mathbb{N} \times \mathbb{N}$, $f = \{(1,2), (2,3), (4,5), (5,6), \ldots\}$

Ex $f: \mathbb{N} \to \mathbb{N}$

$n \mapsto n^2$

$$f = \{(1,1), (2,4), (3,9), (4,16), \ldots\} = \{(n, n^2) \mid n \in \mathbb{N}\}$$

Ex $f: \mathbb{R} \to [-1,1]$

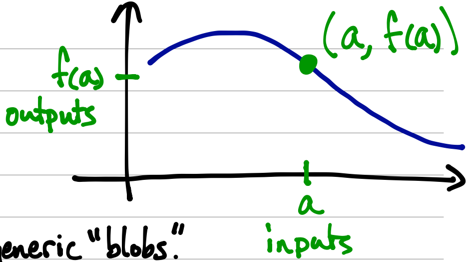
$x \mapsto \cos(x).$

$$f \neq \{(0,1), (0.1, \cos 0.1), (0.01, \cos .01), \ldots, ??\}$$

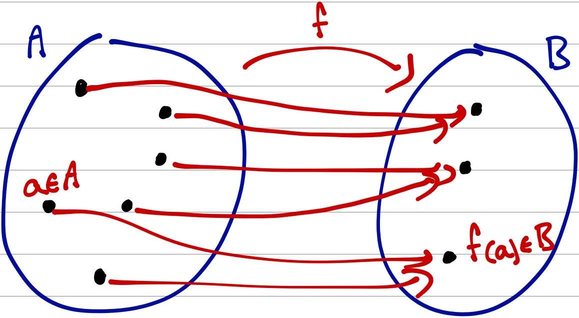
$$= \{(x, \cos x) \mid x \in \mathbb{R}\} \subseteq \mathbb{R} \times [-1,1]$$

You're used to graphing fns:

$(a, f(a)) \subseteq f$



We can also visualize them as generic "blobs."

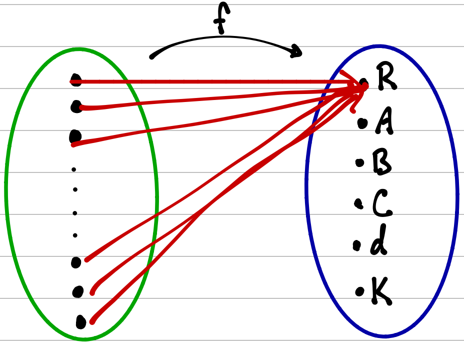


Ex $f: X \to Y$,

X = set of all 3283W students

Y = {Rogness, Albritton, Baker, Corsi, del Mas, Kelley}

$f(x)$ = which instructor is hit by tomato thrown by student x.



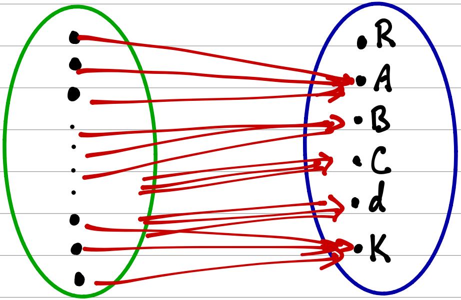
$f(x)$ = person who wrote your exam

Ex $f: X \to Y$,

X = set of all 3283W students

$Y = \{$Rogness, Albritton, Baker, Corsi, del Mas, Kelley$\}$

$f(x)$ = which instructor is hit by tomato thrown by student x.



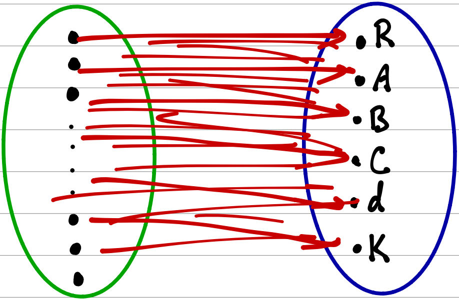
$f(x)$ = person who grades your WQ's

Ex $f: X \to Y$,

X = set of all 3283W students

Y = {Rogness, Albritton, Baker, Corsi, del Mas, Kelley}

$f(x)$ = which instructor is hit by tomato thrown by student x.

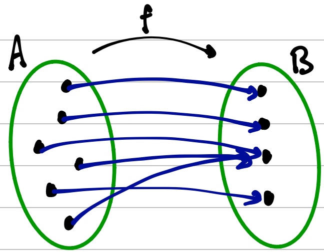


1st six ensure all outputs hit, then each student after makes own choice.

Hugely important properties of fns $f: A \to B$

Def f is surjective (onto, is a surjection) if every potential output in B is an actual output.

(codomain = range(f))

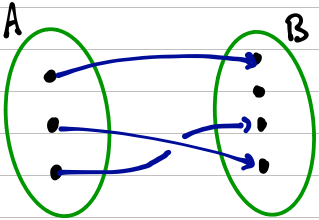


Function notation: $\forall b \in B\ \exists a \in A \ni f(a) = b.$

Relation version: $\forall b \in B\ \exists a \in A \ni (a,b) \in f$

Hugely important properties of fns $f: A \to B$

Def f is injective (1:1, one to one, is an injection) if no two elmts in A are sent to same output in B

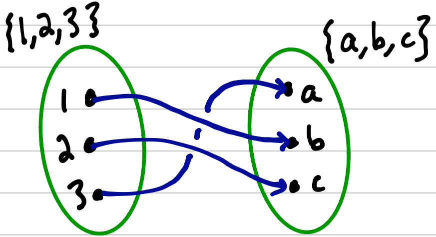


Function notation: If $f(a)=b$ and $f(a')=b$, then $a=a'$.
OR if $a \neq a'$ then $f(a) \neq f(a')$.
(contrapos.)

Relation version: If $(a,b) \in f$ and $(a',b) \in f$ then $a=a'$.

Hugely important properties of fns $f: A \to B$

Def f is bijective (is a bijection, is a 1:1 correspondence) if it is both injective and surjective.



If $\exists$ bijection A to B it means A, B "equivalent" (see §2.4).

They're the same set, with elts relabeled.

Ex $f: \mathbb{R} \to \mathbb{R}$ $\quad f(x) = x^2$

$x \mapsto x^2$

Not surjective – no neg. #'s are outputs. $\forall x \in \mathbb{R},\ f(x) \neq -1.$

Not injective – $f(-2) = f(2) = 4$, but $-2 \neq 2$.

$f: \mathbb{R} \to [0, \infty)$

$x \mapsto x^2$

Is surjective! Let $y \in [0, \infty)$. Then let $x = \sqrt{y} \in \mathbb{R}$, and $f(x) = (x)^2 = (\sqrt{y})^2 = y$.

Not injective - same example.

Note we made function surjective by describing the codomain more accurately – didn't change formula or domain.

$f: [0,\infty) \longrightarrow [0,\infty)$ OR $f: (-\infty, 0] \longrightarrow [0,\infty)$

$x \longmapsto x^2$ $\qquad$ $x \longmapsto x^2$

injective (and still surj) **but** we've changed the fn.

"restricted the domain"

"There are no non-surjective fns. Just poorly defined ones."