<u>Ex</u> if our list looked like this:

$$x_1 = 0.123123123\ldots$$
$$x_2 = 0.33333333\ldots$$
$$x_3 = 0.222222\ldots$$
$$x_4 = 0.98765\ldots$$

$$b = 0.2232\ldots$$

ensures $b \neq x_1$

ensures $b \neq x_2$

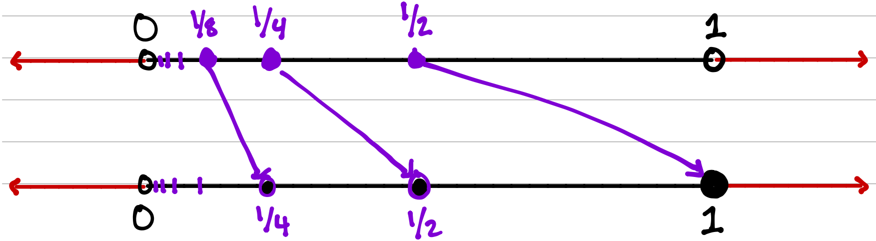
ensures $b \neq x_3$

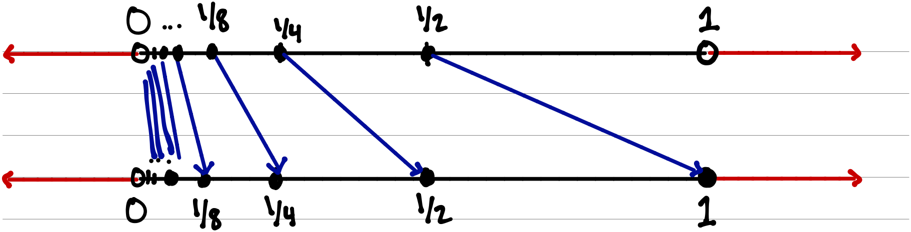
(and so on...)

Another ~~weird~~ cool example $(0,1) \sim (0,1]$

It's not so bad to add one # to a denumerable set and show the result is equinumerous. [ex $f: \mathbb{N}_0 \to \mathbb{N}$, $f(n) = n+1$] It's much harder with an uncountable set.

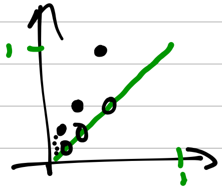
idea: To construct bij'n $f:(0,1) \to (0,1]$, use $f(x) = x$ (id fn) for basically everything. Need to hit 1. Choose to send $1/2 \mapsto 1$. Creates new "gap" in range at $1/2$. Send $1/4 \mapsto 1/2$. (and so on)





Define $f:(0,1)\to(0,1]$ as follows:

$$f(x)=\begin{cases} 2\left(\frac{1}{2^n}\right)=\frac{1}{2^{n-1}}, & x=\frac{1}{2^n}, \text{ some } n\in\mathbb{N} \\ x, & \text{otherwise.}\end{cases}$$



(you check: bij'n $\Rightarrow (0,1)\sim(0,1]$)

Thm TFAE (The following are equivalent)

(a) S is countable

(b) $\exists$ injection $f: S \to \mathbb{N}$

(c) $\exists$ surjection $g: \mathbb{N} \to S$.

$(a) \Leftrightarrow (b)$

$(a) \Leftrightarrow (c)$

$(b) \Leftrightarrow (c)$

usually we prove "just enough" to establish equivalency

$$\begin{tikzcd} (a) \arrow[r, Rightarrow] & (b) \arrow[d, Rightarrow] \\ & (c) \arrow[ul, Rightarrow] \end{tikzcd}$$

or $(a) \Leftrightarrow (b) \Leftrightarrow (c)$

Sketch of Pf of $(a) \Rightarrow (c)$

If S is finite, $S = \{s_1, s_2, \ldots, s_n\}$

define
$$g(1) = s_1$$
$$g(2) = s_2$$
$$\vdots$$
$$g(n) = s_n$$
$$g(m) = s_n, \quad m > n$$

If $S \sim \mathbb{N}$, $S = \{s_1, s_2, s_3, \ldots\}$: $g(n) = s_n$

Putting it all together

If $S \sim T$, they have the same cardinality or cardinal #.
(Remember, this is an equiv. rel'n. [Tu/Th])
So $\{1,2,3\}$, $\{a,b,c\}$ not equal as sets, but have same card'lity.

Def 2.4.15 Denote card'l number of S by $|S|$, so $|S| = |T|$ iff $S \sim T$, i.e.

- Define $|S| \leq |T|$ if $\exists$ injection $S \to T$.
- Define $|S| < |T|$ if $|S| \leq |T|$ but not $|S| = |T|$.

Ex $|\mathbb{N}| \leq |\mathbb{Z}|$: $f : \mathbb{N} \to \mathbb{Z}$, $f(n) = n$ ["inclusion $\mathbb{N} \hookrightarrow \mathbb{Z}$"]
$|\{1,2,3\}| < |\mathbb{N}|$: f inclusion/identity, but $|\{1,2,3\}| \neq |\mathbb{N}|$

The cardinal number or cardinality of a set is (informally) its size:

- cardinal # of $\emptyset$ is $|\emptyset| = 0$.
- cardinal # of $I_n = \{1, 2, \ldots, n\}$ is $|I_n| = n$
- $|\mathbb{N}| = \aleph_0$
- $|\mathbb{R}| = c$ (continuum)

<u>Aside</u>: The Continuum Hypothesis

There is no set whose cardinality is strictly b/w $\aleph_0$ and c.

Thm 2.4.15 Let S, T, U be sets.

(a) $S \subseteq T \Rightarrow |S| \leq |T|$

$f: S \to T$, $x \mapsto x$ } this "identity" or "inclusion" always injective.

$\Rightarrow$ by def^n $|S| \leq |T|$

(b) $|S| \leq |S|$

$S \subseteq S$, use part (a)

(c) $\underbrace{|S| \leq |T|}$ and $\underbrace{|T| \leq |U|} \Rightarrow |S| \leq |U|$

$\exists$ inj $f: S \to T$ $\exists$ inj $g: T \to U$, and $g \circ f: S \to U$ inj^n.

$\Rightarrow |S| \leq |U|$.

(d) $m, n \in \mathbb{N}$, $m \leq n \Rightarrow |I_m| \leq |I_n|$

$I_m = \{1, 2, \ldots, m\} \subseteq \{1, 2, 3, \ldots, n\} = I_n$, use (a).

$\exists$ ∞'ly many ∞'s...

(not P)

Def Given a set S, the power set of S, written $\mathcal{P}(S)$ is the set of all subsets of S.

Ex $S = \{a, b, c, d\}$

$$\mathcal{P}(S) = \{ \underbrace{\emptyset,}_{\text{0 elt subset}} \underbrace{\{a\}, \{b\}, \{c\}, \{d\},}_{\text{1 elt subsets}} \underbrace{\{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\},}_{\text{2 elt subsets}} \underbrace{\{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\},}_{\text{3 elt subsets}} \underbrace{\{a,b,c,d\}}_{\text{4 elt subset}} \}$$

Another way to count # subsets (not list them):

To construct a subset of $\{a,b,c,d\}$...

For each elt, I need to decide whether to include it or not.

(Ex: 4 No's: $\emptyset$; YNNY $\rightsquigarrow \{a,d\}$)

Four Y/N questions $\Rightarrow$ Total possibilities $= 2^4 = 16$

In gen'l for finite S, $|\mathcal{P}(S)| = 2^{|S|}$

Thms
- $|S| < |\mathcal{P}(S)|$
- $|\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$

Corollary $\exists$ infinite "chain" of larger and larger ∞'s:

$$|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < \cdots\cdots$$

A few words concerning §2.5, which covers Axioms of Set Thy.

Math majors are encouraged to read this section, but it's not officially part of the course!

It has the basic Axioms we use to build up set theory and modern mathematics – and possible shortcomings!

(... paradoxes!)