§2.4 Cardinality

Ok, the Hotel Infinity teaches us that infinity is weird. Especially when we compare sizes of infinite sets.

For finite sets, it's easier! If $A = \{1, 2\}$ and $B = \{0, \triangle, \square\}$ then A has 2 elts, B has 3, so B is "larger".

Ex Write these sets in order from "smallest" (i.e. fewest members) to "largest" (most elements).

$\mathbb{N} = \{1, 2, 3, \ldots\}$

$\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$

$\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$

$\mathbb{Q}^+ = \{p/q \geq 0, \text{etc}\}$

$\mathbb{Q}$

all turn out to have same "size"

Irrationals $= \mathbb{R} \setminus \mathbb{Q}$

$\mathbb{R}$

$\mathbb{C}$

$(0,1)$

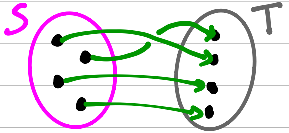
$[0,1)$

$[0,1]$

all same "size" (larger than left column)

BIJECTIONS bring order to this chaos!

Def Two sets S, T are equinumerous if $\exists$ bijection $S \to T$. Write: $S \sim T$.



everything in T is "hit" by exactly one elt in S

idea if $S \sim T$, they're in 1:1 correspondance and are "same set" (with elts relabeled) hence same size

Ex $A=\{1,2,3\}$ and $B=\{a,b,c\}$

$\exists$ bijection $f: A \to B$: $f(1) = b$
$f(2) = a$
$f(3) = c$

f is bij'n "by inspection"

$\{1,2,3\} \sim \{a,b,c\}$

Ex $A=\{1,2,3,4\}$ and $B=\{a,b,c\}$

No bij'n possible so these sets have different "sizes"

Could choose different outputs for $f(1), f(2), f(3)$, but $f(4)$ will break injectivity.

(Google: Pigeonhole Principle)

Ex $A=\{1,2,3,4\}$ and $\mathbb{N}$

No bij'n possible - we can choose up to 4 outputs but will never be surjective.

Ex $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$

$\mathbb{N} = \{1, 2, 3, \dots\}$

$\mathbb{N} \sim \mathbb{N}_0$

$f: \mathbb{N}_0 \longrightarrow \mathbb{N}$, $f(n) = n+1$

Is f surj? Let $m \in \mathbb{N}$. Then $m = f(m-1)$, and $m-1 \in \mathbb{N}_0$.

Is f inj? Let $n \neq m$ in $\mathbb{N}_0$. Then $n+1 \neq m+1$.

Ex $\mathbb{N}, \mathbb{Z}$.

n	$f(n)$
1	0
2	1
3	-1
4	2
5	-2
6	3
$\vdots$	$\vdots$

for $f: \mathbb{N} \to \mathbb{Z}$

$f(n) = (-1)^n \left\lfloor \frac{n}{2} \right\rfloor$ (round down, floor)

bij'n, so $\mathbb{N} \sim \mathbb{Z}$

Ex $\mathbb{N}$, $\mathbb{Q}^+ = \{ \frac{p}{q} \geq 0 \}$ (Cantor)

start

$$
\begin{array}{cccccc}
\frac{0}{1}, & \frac{0}{2}, & \frac{0}{3}, & \frac{0}{4}, & \frac{0}{5}, & \cdots \\
\frac{1}{1}, & \frac{1}{2}, & \frac{1}{3}, & \frac{1}{4}, & \frac{1}{5}, & \cdots \\
\frac{2}{1}, & \frac{2}{2}, & \frac{2}{3}, & \frac{2}{4}, & \frac{2}{5}, & \cdots \\
\frac{3}{1}, & \frac{3}{2}, & \frac{3}{3}, & \frac{3}{4}, & \frac{3}{5}, & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}
$$

We'll construct a bijn $\mathbb{N} \to \mathbb{Q}^+$ as follows

Define $f : \mathbb{N} \to \mathbb{Q}^+$ as

$f(n) = n^{th}$ <u>unique</u> # we meet on this path.

$f(1) = 0 \ (= 0/1)$
$f(2) = 1 \ (= 1/1)$
$f(3) = 2$
$f(4) = 1/2$
$f(5) = 1/3$
$f(6) = 3$
etc.

$\mathbb{N} \sim \mathbb{Q}^+$

Def A set S is...

- **finite** if $S \sim I_n = \{1, 2, 3, \ldots, n\}$
- **denumerable** if $S \sim \mathbb{N}$
- **countable** if S is finite or denumerable ⚠ two cases!

- **uncountable** if S is not countable.

$\exists$ handy-dandy Venn Diagram in your book (p84)