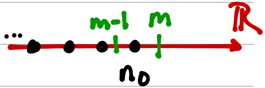
Thm 3.3.9 Archimedean Property of $\mathbb{R}$

$\mathbb{N}$ has no upper bound in $\mathbb{R}$.



Pf By contradiction. Suppose $\mathbb{N}$ is bdd above, $\mathbb{N} \neq \phi \Rightarrow \exists$ least upper bd, $m = \sup \mathbb{N}$. Then $m-1$ not upper bd $\Rightarrow \exists\, n_0 \in \mathbb{N}$, $n_0 > m-1$. Then $n_0 + 1 > m$, and $n_0 + 1 \in \mathbb{N}$, m not upper bd $\lightning$.

Thm 3.3.10 TFAE

(*) Archimedean Property

(a) $\forall z \in \mathbb{R}, \exists n \ni n > z$.

(b) $\forall x > 0\ \forall y \in \mathbb{R}\ \exists n \in \mathbb{N} \ni nx > y$

(c) $\forall x > 0\ \exists n \in \mathbb{N} \ni 0 < \frac{1}{n} < x$.

Pf (*) $\Rightarrow$ (a) Assume not, so $\exists z$ s.t $\forall n$, $z \geq n$. $\lightning$ (bd for $\mathbb{N}$)

(a) $\Rightarrow$ (b) Choose $z = y/x$ in (a)

(b) $\Rightarrow$ (c) Choose $y = 1$ in (b).

(Also (c) $\Rightarrow$ (*)?)

Finally...

Thm $\mathbb{Q}$ is **dense** in $\mathbb{R}$: $\forall\, x, y \in \mathbb{R}$ with $x<y$, $\exists\, r \in \mathbb{Q} \ni x<r<y$.

Proof: See book.

Constructive Method: Write out decimal exp's of x, y:

$$x = 3.1415692\ldots$$
$$y = 3.1412987\ldots$$

1st time they differ, "split the difference"

$$r = 3.1413 = \frac{31413}{10{,}000}$$

Also...

Thm 3.3.15 $\forall x, y \in \mathbb{R}$ $(x<y)$, $\exists$ irrational # $w \in \mathbb{R} \setminus \mathbb{Q}$ s.t. $x < w < y$.

Pf $\exists$ r $(\neq 0)$ in $\mathbb{Q}$ between $\frac{x}{\sqrt{2}}$, $\frac{y}{\sqrt{2}}$:

$$\frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}}$$

$$x < \underbrace{r\sqrt{2}}_{\text{irr'l}} < y$$