§3.4 Topology of $\mathbb{R}$

In this context, "topology" refers to open and closed sets in $\mathbb{R}$. We'll also discuss various "parts" of any $S \subseteq \mathbb{R}$.

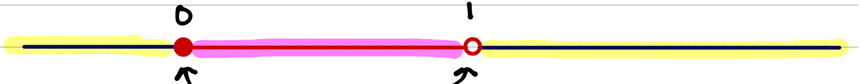
With sequences, we talk about "points close to x." This section lays groundwork for putting "close to" on a rigorous footing:

☆☆ • neighborhoods
☆ • interior (pts), boundary (pts)
☆☆☆ • open, closed set in $\mathbb{R}$.

[• accumulation pts, closure]

Foreshadowing: we'll learn the following parts of a set $S \subseteq \mathbb{R}$:

$[0,1) \subseteq \mathbb{R}$

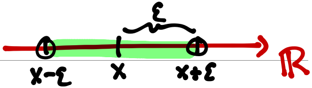


boundary points of $[0,1)$

= interior of $[0,1)$

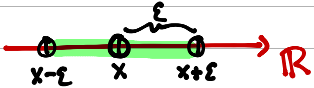
= exterior of $[0,1)$

<u>Def</u> Let $x \in \mathbb{R}$, $\varepsilon > 0$. Then



$$N(x;\varepsilon) = \{y : |y-x| < \varepsilon\}$$

is the neighborhood (nbhd) centered at x with radius ε.
Also, ∃ punctured nbhd: $N^*(x;\varepsilon) = \{y \mid 0 < |y-x| < \varepsilon\}$



<u>Equivalently</u>,

$$N(x;\varepsilon) = (x-\varepsilon, x+\varepsilon)$$

$$N^*(x;\varepsilon) = (x-\varepsilon, x) \cup (x, x+\varepsilon)$$

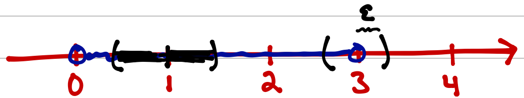
$N(x;\varepsilon) = \{ |y-x| < \varepsilon \}$. $\qquad N^*(x;\varepsilon) = \{ 0 < |y-x| < \varepsilon \}$

Ex $N(2;1) = (1,3)$ $\qquad N^*(10;4) = (6,10) \cup (10,14)$

$N(0;1/2) = (-1/2, 1/2)$

Def $x \in S \subseteq \mathbb{R}$ is an ***interior point of S*** if $\exists\, \varepsilon > 0$ such that $N(x;\varepsilon) \subseteq S$.

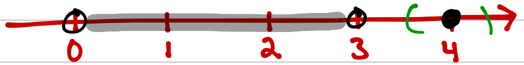
Ex $(0,3)$



$x = 1$ is int pt of S: $N(1;1/2) = (1/2, 3/2) \subseteq S$

$x = 3$ is <u>not</u> int pt of S: $\forall\, \varepsilon > 0$, $N(3;\varepsilon) = (3-\varepsilon, 3+\varepsilon)$ includes #'s ≥ 3, hence $N(3;\varepsilon) \not\subseteq S$.

Def Conversely, if every nbhd of x contains pts in S ($N \cap S \neq \emptyset$) and points not in S ($N \cap S^c \neq \emptyset$) then x is a boundary point of S.

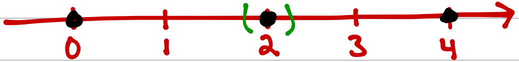


Ex Argument from prev slide says 3 is bdy pt of $(0,3)$;
Any $N(3;\varepsilon)$ will contain #'s less than 3 and in S,
but also #'s ≥ 3 and not in S.

$N(4;2) = (2,6)$ contains pts in $(0,3]$, pts not in $(0,3)$, but 4 not bdy pt because this doesn't happen for every nbhd. $(N(4;1/2) \cap (0,3) = \emptyset)$

Ex Find int and bd for these sets:

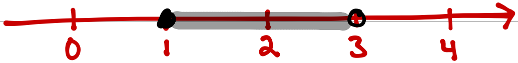
$S = \{0, 2, 4\}$



int S = set of int. pts = $\emptyset$

bd S = set of bound pts = $\{0, 2, 4\}$

$T = [1, 3)$



int $S = (1, 3)$

bd $S = \{1, 3\}$

Open / Closed Sets

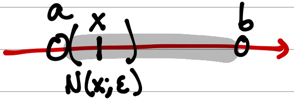
⚠ Our approach will be slightly different than the book's. Everything will work out the same in the end, but our def's are the books thms, and vice versa.

Def $S \subseteq \mathbb{R}$ is <u>open</u> in $\mathbb{R}$ if every $x \in S$ is an interior point.

- equivalently, $S \subseteq \operatorname{int} S$.
- since $\operatorname{int} S \subseteq S$ always, could also say $S = \operatorname{int} S$.

Examples

① Any inverval $(a,b) = \{a < x < b\}$ is open set (hence name "open interval").



Let $d_1 = |x-a|$, $d_2 = |x-b|$. If $\varepsilon = \min\{d_1, d_2\}$, then $N(x;\varepsilon) \subseteq (a,b)$

② $N(x;\varepsilon) = (x-\varepsilon, x+\varepsilon)$

This is an open set by ① — "open" nbhds.

③ $\mathbb{R}$ open

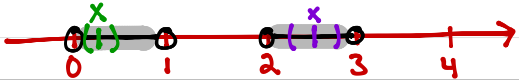
Let $x \in \mathbb{R}$. $\forall \varepsilon > 0$ (not just one), $N(x;\varepsilon) \subseteq \mathbb{R}$.

(4) $\emptyset \subseteq \mathbb{R}$ open for trivial reasons.

if $x \in \emptyset$, then $x \in \operatorname{int} \emptyset$

Always F, so impl'n true $\Rightarrow$ $\emptyset$ open

(5) $S = (0,1) \cup (2,3)$



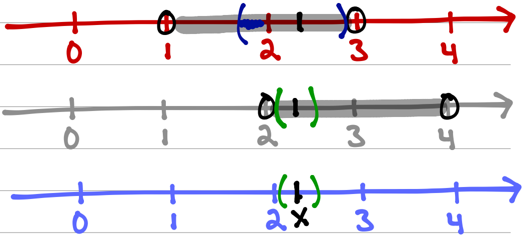
Let $x \in (0,1) \cup (2,3)$. Then $x \in (0,1)$ or $x \in (2,3)$.

If $x \in (0,1)$, which is an open set, then $\exists\, N(x;\varepsilon_1) \subseteq (0,1) \subseteq S$

If $x \in (2,3)$, which is an open set, then $\exists\, N(x;\varepsilon_2) \subseteq (2,3) \subseteq S$

6 $S=(1,3)\cap(2,4)$

[$=(2,3)$, open]



Let $x\in(1,3)\cap(2,4)$, so $x\in(1,3)$ <u>and</u> $x\in(2,4)$

Because $x\in(1,3)$, $\exists\ \varepsilon_1>0$ s.t. $N(x;\varepsilon_1)\subseteq(1,3)$

Because $x\in(2,4)$, $\exists\ \varepsilon_2>0$ s.t. $N(x;\varepsilon_2)\subseteq(2,4)$

<u>Key</u> if we choose $\varepsilon=\min\{\varepsilon_1,\varepsilon_2\}$, then $N(x;\varepsilon)$ is guaranteed to be in both $(1,3)$ <u>and</u> $(2,4)$, hence $(1,3)\cap(2,4)$

More generally,

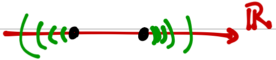
Thm 3.4.10

(a) Any union of open sets is open.
↳ finitely or infinitely many!

(b) An intersection of finitely many open sets is open.

Ex $\displaystyle\bigcup_{n=1}^{\infty}(0,n) = (0,1)\cup(0,2)\cup(0,3)\cup\cdots = (0,\infty)$ open

$\displaystyle\bigcap_{n\in\mathbb{N}}\left(-\tfrac{1}{n},1+\tfrac{1}{n}\right) = (-1,2)\cap\left(-\tfrac{1}{2},\tfrac{3}{2}\right)\cap\left(-\tfrac{1}{3},\tfrac{4}{3}\right)\cap\cdots = \underbrace{[0,1]}_{\text{not open}}$



not open
(0 and 1 not int. pts)

Def $S \subseteq \mathbb{R}$ is closed if $\mathbb{R} \setminus S = S^c$ is open

⚠ open, closed not opposites!!

$[0,1)$ is neither:

not open b/c 0 not int pt.

not closed b/c $[0,1)^c = (-\infty, 0) \cup [1, \infty)$

not open - 1 not an int pt!

A Topologist as Storekeeper...

Examples

① $S = [0,1]$

$S^c = (-\infty, 0) \cup (1, \infty)$ union of two open sets is open.

S^c open $\Rightarrow$ S closed.

② $\emptyset$

$\emptyset^c = \mathbb{R} \setminus \emptyset = \mathbb{R}$ open $\Rightarrow$ $\emptyset$ closed!

③ $\mathbb{R}$

$\mathbb{R}^c = \mathbb{R} \setminus \mathbb{R} = \emptyset$ open $\Rightarrow$ $\mathbb{R}$ closed!

$\emptyset$, $\mathbb{R}$ are both open <u>and</u> closed (!!) : <u>CLOPEN</u>.


Yes we're
CLOPEN

$\exists$ different characterization:

Thm $S \subseteq \mathbb{R}$ is closed iff it contains all its bdy pts

($\text{bd}\, S \subseteq S$)

Ex $[a,b]$: $[a,b]^c = (-\infty, a) \cup (b, \infty)$ open $\Rightarrow [a,b]$ closed.

OR

$\text{bd}[a,b] = \{a,b\} \subseteq [a,b] \Rightarrow [a,b]$ closed.

$\{0,2,4\}$: $\text{bd}\{0,2,4\} = \{0,2,4\} \subseteq \{0,2,4\} \Rightarrow$ closed

Thm (a) Any (finite or infinite) intersection of closed sets is closed.

(b) Any finite union of closed sets is closed.

Pf (b) Suppose $A_1, A_2, \dots, A_n$ are closed, so $A_i^c = \mathbb{R} \setminus A_i$ is open.

To check if $A_1 \cap \cdots \cap A_n$ is closed, see if its compl't is open:

$$(A_1 \cap \cdots \cap A_n)^c = A_1^c \cup A_2^c \cup \cdots \cup A_n^c$$

which is a union of open sets, hence open.

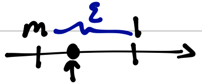
Test announcements:

* Study guide to be posted
* Harder than Exam I (by nature of the material)
* You can expect a bit from §3.4 and §4.1 — but in some reasonably expected way.

⚠ Many problems involve the same alg.

Ex $A = \left\{ \frac{n}{n+1} : n \in \mathbb{N} \right\} = \left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots \right\}$

Heart of pf that $\sup A = 1$ amounts to showing:
$\exists\, n \ni \frac{n}{n+1} > m.$



That's similar to showing $1 \in \operatorname{bd} A$, or $\frac{n}{n+1} \to 1$.

<u>Not</u> in course, but you can read about accumulation pts, <u>closure</u> of a set.

Also, §3.5 (compact sets) not in course.... but:

<u>Def</u> A set $S \subseteq \mathbb{R}$ is <u>compact</u> if it is <u>closed</u> and <u>bounded</u> (both above and below)

(not <u>actual</u> def – Heine-Borel Thm)