§4.2 Limit Laws

A section of time-saving tools.

Goal: Recover what we "know" about seq's and limits from calculus — i.e. show that they follow from our rigorous ε-N def$^{\underline{n}}$...

Thm 4.2.1 Suppose $s_n \to s$, $t_n \to t$, $k \in \mathbb{R}$. Then

(a) $\lim(s_n + t_n) = s + t = \lim s_n + \lim t_n$ $[s_n + t_n \to s + t]$

(b) $\lim(k s_n) = k \cdot s = k \cdot \lim s_n$

$\lim(k + s_n) = k + s = k + \lim s_n$

(c) $\lim(s_n t_n) = s \cdot t = (\lim s_n) \cdot (\lim t_n)$

(d) $\lim\left(\frac{s_n}{t_n}\right) = \frac{s}{t} = \frac{\lim s_n}{\lim t_n}$

provided $t \neq 0$ and $t_n \neq 0$ $\forall n$.

can work around a bit.

Thm 4.2.1 Makes life much easier

Ex Prove $\frac{n^2+2n}{n^3-5} \longrightarrow 0$

$$\lim \frac{n^2+2n}{n^3-5} \neq \frac{\lim n^2+2n}{\lim n^3-5}$$ (b/c these limits d.n.e.)

(algebra) $\|$

$$\lim \frac{1/n + 2/n^2}{1-5/n^3} \overset{\substack{4.2.1\\(d)}}{=} \frac{\lim 1/n + 2/n^2}{\lim 1-5/n^3}$$

$$\overset{\substack{4.2.1\\(a)}}{=} \frac{\lim 1/n + \lim 2/n^2}{\lim 1 - \lim 5/n^3}$$

HW: $\frac{c}{n^k} \longrightarrow 0$

$$= \frac{0+0}{1-0} = \frac{0}{1} = 0$$

Pf of (a) $[s_n \to s,\ t_n \to t] \Rightarrow (s_n + t_n) \to s + t$

Let $\varepsilon > 0$. We need to make $|(s_n + t_n) - (s+t)| < \varepsilon$.

Using Δ-ineq, $|(s_n - s) + (t_n - t)| \le |s_n - s| + |t_n - t|$

Because $s_n \to s$, $t_n \to t$, there exist:

N_1 s.t. $n > N_1$, $|s_n - s| < \varepsilon/2$
N_2 s.t. $n > N_2$, $|t_n - t| < \varepsilon/2$

If $n > N = \max\{N_1, N_2\}$, then both ineqs are true, and

$$\begin{aligned} |(s_n + t_n) - (s+t)| = |(s_n - s) + (t_n - t)| &\le |s_n - s| + |t_n - t| \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon \end{aligned}$$

What if, instead of $\frac{1}{n}, \frac{1}{n^2}, \frac{2n^3-3}{n^4+6}$ we have sequences with

factorials: $n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$

or exponentials: e^n, 2^n, 3^n, etc. $\qquad a_n = \frac{n!}{2^n}$

Thm 4.2.7 "Ratio Test for Sequences"

Suppose $s_n \geq 0$, ratios $\left(\frac{s_{n+1}}{s_n}\right) \to L$.
If $L < 1$, then $s_n \to 0$.

Think positive terms, $\underbrace{\text{getting smaller}}_{\frac{s_{n+1}}{s_n} < 1}$, so $s_n \to 0$.

If $s_n \geq 0$, $\frac{s_{n+1}}{s_n} \to L < 1$, then $s_n \to 0$.

<u>Ex</u> $a_n = \frac{1}{2^n}$ $\qquad \frac{a_{n+1}}{a_n} = \frac{1/2^{n+1}}{1/2^n} = \frac{1}{2^{\cancel{n+1}}} \cdot \frac{\cancel{2^n}}{1} = \frac{1}{2} < 1 \Rightarrow a_n \to 0.$

$b_n = \frac{2^n}{n!}$ $\qquad \frac{b_{n+1}}{b_n} = \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} = \frac{2^{\cancel{n+1}} \cancel{n(n-1)\cdots 2 \cdot 1}}{\cancel{2^n} \cdot (n+1)\cancel{n(n-1)\cdots 1}}$

$= \frac{2}{n+1} \to 0 < 1 \Rightarrow b_n \to 0.$

$c_n = \frac{1}{n}$ $\qquad \frac{c_{n+1}}{c_n} = \frac{1}{n+1} \cdot \frac{n}{1} = \frac{n}{n+1} \to 1$

⚠ Thm 4.2.7 tells us <u>nothing</u> about convergence of c_n. We <u>know</u> $c_n \to 0$, but we must show a different way.

$\exists$ other thms in this section - you read.

Thm 4.2.4 (Another "Squeeze" Thm)
Suppose $s_n \to s$, $t_n \to t$. If $s_n \le t_n$ $\forall n$, then $s \le t$.

Corollary 1 with same setup and $0 \le s_n \le t_n$, $t_n \to 0 \Rightarrow s = 0$

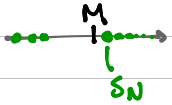
Corollary 2 (set all $s_n = 0$, so $s = 0$) $\Rightarrow t \geq 0$

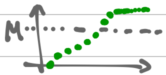
Infinte Limits

$s_n = n$ diverges but for other reasons than $(t_n) = (-1, 1, -1, \ldots)$

It's useful to be able to describe this behavior.

$\forall M \in \mathbb{R} \; \exists N$ s.t. $n > N \Rightarrow s_n > M$.





Similarly, $s_n \to -\infty$ (diverges!)

$\forall M \in \mathbb{R} \; \exists N$ s.t. $n > N \Rightarrow s_n < M$

$s_n \to +\infty$ if: $\forall M \in \mathbb{R}\ \exists N$ s.t. $n > N \Rightarrow s_n > M$

Ex Prove $s_n = n^2$ diverges to $+\infty$

T: Want to find n s.t. $n^2 > M$ for any M.

So $n > \sqrt{M}$ if $M \geq 0$. (So use $N = \sqrt{M}$)

If $M < 0$, then $n^2 > M$ always (so can use $N = 0$)

P: $\forall M \in \mathbb{R}$, choose $N = \sqrt{M}$ if $M \geq 0$, or else $N = 0$ if $M < 0$.

Then $n > N \Rightarrow n^2 > N^2 = M$ if M positive,

or $n^2 > M$ (always) if $M < 0$.

You Read

Thm 4.2.12 Suppose $s_n \le t_n \ \forall n \in \mathbb{N}$

(a) $s_n \to +\infty \Rightarrow t_n \to +\infty$

(b) $t_n \to -\infty \Rightarrow s_n \to -\infty$

Thm 4.2.13 $s_n > 0$, $s_n \to +\infty \Rightarrow \frac{1}{s_n} \to 0$

(You read)