Recall / Announcements.

HW posted later today

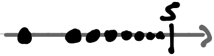
No quiz this week!

Wednesday: Work through whichever HW prob's are trickiest – then fun stuff.

☆ We'll skip to Chapter 8 (Series) today, then retreat to Chapter 5 in December.

Cauchy Sequences

So far we've described convergence as elts of a sequence (eventually) bunching up next to a limit.



<u>Def</u> A seq (s_n) of real #'s is a <u>Cauchy Sequence</u> if

$$\forall \varepsilon > 0 \ \exists N \text{ s.t. } n, m > N \Rightarrow |s_n - s_m| < \varepsilon$$

i.e. eventually the #'s bunch up together

Leibniz

Newton

Cauchy

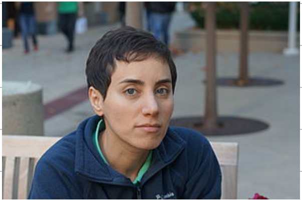
Weierstrass

2014 Fields Medalists

Artur Avila

Manjul Bhargava

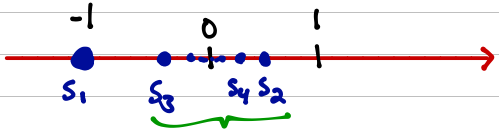
Martin Hairer



Maryam Mirzakhani

Cauchy: $\forall \varepsilon>0\ \exists N$ s.t. $n,m>N \Rightarrow |s_n - s_m| < \varepsilon$

Ex $s_n = \dfrac{(-1)^n}{n}$



$\forall \varepsilon>0$, we know $\exists N$
s.t. $0<\frac{1}{N}<\varepsilon/2$.

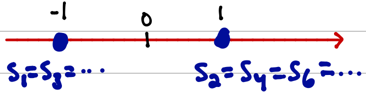
dist. of $\frac{1}{2} - (-\frac{1}{3}) = \frac{5}{6}$.
For $\varepsilon = 5/6$, $N=3$,
$n,m>N \Rightarrow |s_n-s_m|<\varepsilon$.

Then $n,m>N$ (so both $\frac{1}{n}, \frac{1}{m} < \varepsilon/2$) $\Rightarrow$

$$|s_n-s_m| \le |s_n|+|s_m| = \left|\frac{(-1)^n}{n}\right| + \left|\frac{(-1)^m}{m}\right| = \frac{1}{n}+\frac{1}{m}<\varepsilon.$$

Cauchy: $\forall \varepsilon > 0\ \exists N$ s.t. $n, m > N \Rightarrow |s_n - s_m| < \varepsilon$

Ex $t_n = (-1)^n = \begin{cases} -1, & n \text{ odd} \\ 1, & n \text{ even} \end{cases}$



This is not a Cauchy sequence. We can't force the #'s to bunch up as needed in def$^{\underline{n}}$

Not Cauchy: $\exists \varepsilon > 0$ s.t. $\forall N$, $\exists\ n, m > N$ and $|s_n - s_m| \geq \varepsilon$.

Say $\varepsilon = 1$. For any N, we can always find $n, m > N$ with n odd, m even

$$\Rightarrow |s_n - s_m| = |(-1) - 1| = |-2| = 2 > \varepsilon.$$

Cauchy: $\forall \varepsilon > 0 \ \exists N$ s.t. $n,m > N \Rightarrow |s_n - s_m| < \varepsilon$

Why do we care?

Thm (s_n) converges $\Leftrightarrow$ (s_n) Cauchy

Pf $\Leftarrow$ not in this course.

$\Rightarrow$ Suppose $s_n \to s$, let $\varepsilon > 0$ be given.
Must show $\exists N$ s.t. $n,m > N \Rightarrow |s_n - s_m| < \varepsilon$.

We know $\exists N$ s.t. $n > N \Rightarrow |s_n - s| < \varepsilon/2$.
Then $n,m > N$ gives

$$|s_n - s_m| = |s_n - s - s_m + s| = |s_n - s - (s_m - s)|$$
$$\leq |s_n - s| + |s_m - s| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$