§4.3 Monotone and Cauchy Sequences

More ways to prove sequences converge without resorting to ε's and N's .... as much.

Concepts:

★★★ 1. Monotone Convergence Thm (MCT)

★ 2. ... Applications to certain recursive seq's.

★★ 3. Cauchy Sequences

Def s_n is <u>increasing</u> if $s_n \le s_{n+1}$ $\forall n$
or <u>decreasing</u> if $s_n \ge s_{n+1}$ $\forall n$

With $<, >$ instead of $\le, \ge$, we'd say

strictly incr'g, decr'g

Allowing for equality gives us monotonically incr'g, decr'g. (default)

(s_n) <u>monotone</u> $\iff$ (monotonically) incr'g or decr'g.

Examples of Seq's which are...

Strictly Increasing $a_n = n$ $c_n = n+1$

$b_n = 2^n$

Strictly Decreasing $a_n = -n$ $b_n = 2^{-n}$

Mon. (not strictly) Incr'g: $(a_n) = (1, 1, 2, 3, 4, 5, 6 \ldots)$

$(b_n) = (1, 1, 2, 3, 5, 8, 13, \ldots)$

$(c_n) = (\lfloor n/2 \rfloor) = (0, 1, 1, 2, 2, 3, 3, \ldots)$

Both Incr'g and Decr'g

$(a_n) = (1, 1, 1, 1, 1, 1, \ldots)$

Not monotone $a_n = \sin n$

$(b_n) = (-1, 0, 1, 0, -1, 0, 1, 0, -1, \ldots)$

How can we show (a_n) is increasing?

1. Directly with Algebra: Show $a_n \le a_{n+1} \ \forall n$

OR $\frac{a_{n+1}}{a_n} = \cdots = \cdots \ge 1$ (if $a_n > 0$)

OR $a_{n+1} - a_n = \cdots = \cdots \ge 0$

Ex $\frac{n-1}{n}$ is incr'g:

avoid
$$\frac{n-1}{n} \le \frac{(n+1)-1}{n+1}$$
$$(n-1)(n+1) \le n^2$$
$$n^2 - 1 \le n^2$$
$$-1 \le 0$$

Better $a_{n+1} - a_n = \frac{n}{n+1} - \frac{n-1}{n} = \frac{n^2 - n^2 + 1}{(n+1)(n)} = \frac{1}{n^2+n} \ge 0 \ \forall n$

2. By induction.

Show $a_2 \geq a_1$. Assume $a_k \geq a_{k-1}$, use that to prove $a_{k+1} \geq a_k$.

3. <u>Calculus</u>

<u>Ex</u> $f(n) = n^3$, which is "really" just $g: \mathbb{R} \to \mathbb{R}$, $g(x) = x^3$ where we've restricted domain to $\mathbb{N}$ instead of $\mathbb{R}$.

$g'(x) = 3x^2 > 0 \Rightarrow g$ is (strictly) increasing, i.e. $\forall x < y$, $g(x) < g(y)$. In particular, true for $x = n$, $y = n+1$....

(Probably best to ask <u>before</u> using this in 3283W...)

Monotone Convergence Thm (4.3.3)

A monotone sequence is convergent iff it's bounded.

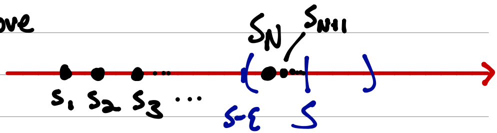
Pf Let s_n be monotone. If s_n converges, then it's bounded. (Thm at end of 4.1. True for all conv. seq's).

In other dir'n, suppose s_n bdd and show it converges.

We'll do case with s_n incr'g and bdd.

(decr'g and bdd similar).

Suppose s_n incr'g and bdd above



By completeness axiom, $\{s_n\}$ has a least upper bound; write $s = \sup\{s_n\}$

I claim $s_n \to s$. Let $\varepsilon > 0$. Need to show $\exists\, N$ s.t. $n > N \Rightarrow |s_n - s| < \varepsilon$.

Since $s = \sup\{s_n\}$, $s-\varepsilon$ <u>NOT</u> upper bound $\Rightarrow \exists\, N$ s.t. $s_N > s-\varepsilon$. Because s_n incr'g, for <u>all</u> $n > N$

$$s-\varepsilon < s_N \le s_n \le s$$

$$\Rightarrow |s_n - s| < \varepsilon, \text{ as desired}$$

MCT VERY useful - we can often prove $a_n \to a$ w/o ε's, N's.

Ex $(a_n) = \left(1 - \frac{1}{n}\right) = \left(\frac{n-1}{n}\right) = \left(0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\right)$

We've shown this is bdd below by 0: $a_n \geq 0$
and bdd above by 1: $a_n \leq 1$. $\Big\}$ §3.3

Last time: a_n incr'g.

$\Rightarrow$ a_n converges.

Ex $s_1 = 1$, $s_{n+1} = \sqrt{1+s_n}$

$(s_n) = \left(1, \sqrt{1+1}, \sqrt{1+\sqrt{2}}, \sqrt{1+\sqrt{1+\sqrt{2}}}, \dots\right)$

1 1.414... 1.5537... 1.59805...

Claim 1 s_n incr'g. $s_2 = \sqrt{2} \geq 1 = s_1$. Now assume $s_k \geq s_{k-1}$. Then

$$s_{k+1} = \sqrt{1+s_k} \geq \sqrt{1+s_{k-1}} = s_k$$

Claim 2 s_n bdd below by $s_1 = 1$ b/c s_n is incr'g. Now prove s_n is bounded above: show $s_n \leq 2$ for all n.

$s_1 = 1 \leq 2$. Now suppose $s_k \leq 2$. Then

$$s_{k+1} = \sqrt{1+s_k} \leq \sqrt{1+2} = \sqrt{3} \leq 2.$$

$\Rightarrow s_n$ converges by MCT.

Ok, great – $s_1 = 1$, $s_{n+1} = \sqrt{1+s_n}$ converges – but to what?

Key for any sequence $\lim_{n\to\infty} s_n = \lim_{n\to\infty} s_{n+1}$.

Ex $\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots\right)$ $\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \ldots\right)$ } converge to same #.

Thus $\lim s_n = \lim s_{n+1}$

$\lim s_n = \lim \sqrt{1+s_n}$ (Assume $s_n \to s$)

$s = \sqrt{1+s}$

$s^2 = 1+s$

$s^2 - s - 1 = 0$ Solve (get φ)