Ex In first example, $\underline{1-1}\,\underline{+1}-1+1-1+\cdots$

$s_1=1,\ s_2=0,\ s_3=1;\ (s_n)=(1,0,1,0,1,\ldots)$ diverges

Ex $\displaystyle\sum_{n=0}^{\infty}\frac{1}{2^n}=1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots=\lim\frac{1-(1/2)^{n+1}}{1-1/2}=\frac{1-0}{1/2}=2.$

From HW: $\displaystyle 1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^n}=\frac{1-(1/2)^{n+1}}{1-1/2}$

⚠ $\displaystyle\sum_{n=0}^{\infty}r^n=\frac{1}{1-r}$ for $|r|<1$ (based on HW)

Ex $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n\cdot 2}{3^n}=\sum_{n=0}^{\infty}2\left(\frac{-1}{3}\right)^n=2\sum_{n=0}^{\infty}\left(-\frac{1}{3}\right)^n=\frac{2\cdot 1}{1-(-1/3)}=\frac{2\cdot 1}{4/3}=6/4.$

$=3/2.$

Thm

Thms in §§4.1–4.3 all apply to series (by appling them to their sequences of partial sums.)

<u>Thm 8.1.6</u> $\sum a_n$ converges $\left( \Leftrightarrow s_n = \sum_{k=1}^{n} a_k \text{ converges} \right) \Leftrightarrow s_n$ Cauchy

<u>Thm 8.1.4</u> If $\sum a_n = s$, $\sum b_n = t$, then

(a) $\sum (a_n + b_n) = \sum a_n + \sum b_n = s + t$

(b) $\sum k a_n = k \sum a_n \quad \forall k \in \mathbb{R}$.

<u>Pf of (a)</u> (Sketch) $\sum a_n = s \Leftrightarrow$ partial sums $s_n \to s$

$\sum b_n = t \Leftrightarrow$ partial sums $t_n \to t$

partial sums of $\sum (a_n + b_n)$ are $u_n = a_1 + b_1 + a_2 + b_2 + \cdots + a_n + b_n = s_n + t_n$

By Thm 4.2.1(a) $\lim (s_n + t_n) = \lim s_n + \lim t_n = s + t$.

⚠ Be very careful with thms involving seq's/series — converses are often NOT true!

Ex $a_n = 1$, $\sum a_n = 1+1+1+\cdots = +\infty$
$b_n = -1$, $\sum b_n = -1-1-1-\cdots = -\infty$ } diverge

$\sum(a_n+b_n) = \sum(1-1) = \sum 0 = 0+0+0+\cdots = 0$

DO NOT split $\sum(a_n+b_n)$ into $\sum a_n + \sum b_n$ unless you know the separate pieces converge.

Obligatory (and important) Example "The Harmonic Series"

$$\sum_{n=1}^{\infty} \frac{1}{n} = \underline{1+\frac{1}{2}+\frac{1}{3}}+\frac{1}{4}+\cdots = +\infty \;(!!)$$

$S_1 = 1$

$S_2 = 1 + 1/2 = 3/2$

$S_3 = 1 + \frac{1}{2} + \frac{1}{3} = \frac{9}{6} + \frac{2}{6} = \frac{11}{6}$

$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = ?$

$S_n$ increasing, $S_n$ hard to find

We know: $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$ converges $\Leftrightarrow$ $s_n$ Cauchy:

$\forall \varepsilon > 0\ \exists N$ s.t. $n, m > N \Rightarrow |s_n - s_m| < \varepsilon$

We'll show $s_n$ not Cauchy $\Leftrightarrow$ $s_n$ diverges $\overset{\text{(def)}}{\Leftrightarrow}$ $\sum \frac{1}{n}$ diverges.

Suppose $m > n > 0$, so $\frac{1}{m} < \frac{1}{n}$, $\frac{1}{m} < \frac{1}{n+1}$, $\frac{1}{m} < \frac{1}{n+2}, \dots, \frac{1}{m} < \frac{1}{m-1}$

$$s_m - s_n = \left(1 + \frac{1}{2} + \dots + \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{m}\right) - \left(1 + \frac{1}{2} + \dots + \frac{1}{n}\right)$$

$$= \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{m}$$

$$\geq \underbrace{\frac{1}{m} + \frac{1}{m} + \dots + \frac{1}{m}}_{m-n} = \frac{m-n}{m} \qquad \Rightarrow s_m - s_n \geq \frac{(m-n)}{m} = 1 - \frac{n}{m}$$

If $\varepsilon = 1/2$, for any $N$, choose $m, n > N$ with $m = 3n \Rightarrow s_m - s_n = 1 - \frac{1}{3}$

$= 2/3 \geq \varepsilon$.

In other words, given <u>any</u> $M \in \mathbb{R}$, $\exists$ $N$ s.t. $1 + \frac{1}{2} + \cdots + \frac{1}{N} > M$

<u>BUT</u> $S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ increases so slowly that the #'s involved are <u>ginormous</u>!

| $n$ | $S_n$ | $\gamma + \ln n$ |
|---|---|---|
| 1 | 1 | 0.577... |
| 2 | 1.5 | 1.27 |
| 10 | 2.93 | 2.88 |
| 100 | 5.187 | 5.182 |
| 1000 | 7.485 | 7.48497 |

($\leftarrow$ where $1 + \frac{1}{2} + \cdots + \frac{1}{n} \approx \gamma + \ln n$, $\gamma \approx 0.577$)

So to get $1 + \frac{1}{2} + \cdots + \frac{1}{n} > M$,

solve $\ln n + \gamma > M$

$$\ln n > M - \gamma$$

$$n > e^{M-\gamma}$$

$$M = 100 \Rightarrow n > 10^{43}$$

$$M = 1000 \Rightarrow n > 10^{434}$$

$$M = 1\,000\,000 \Rightarrow n > 10^{434\,290}$$

An aside - does anything diverge more slowly?

$$\sum_{p \text{ prime}} \frac{1}{p} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} = +\infty \quad \text{(tricky)}$$

And yet,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}$$

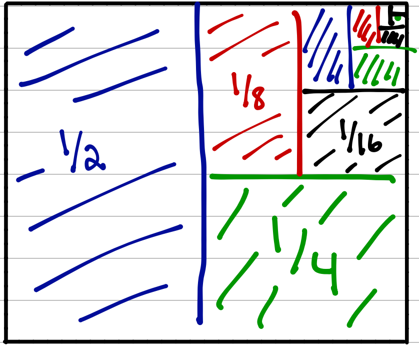
$$\sum_{n=1}^{\infty} \frac{1}{n^4} = 1 + \frac{1}{16} + \frac{1}{81} + \frac{1}{256} + \cdots = \frac{\pi^4}{90} \quad (\leftarrow ?)$$

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = 1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \cdots = \zeta(3) \approx 1.20205\ldots$$

(irrational; Apery)

# Aside #2: Geometric Representation of Series

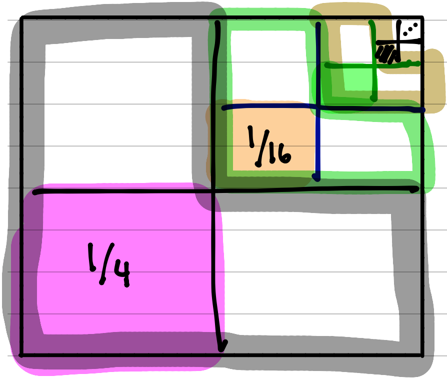
Unit Square:




$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots=1$$

# Aside #2: Geometric Representation of Series

Unit Square:




Shade 1/3 of each piece.

$$\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256}+\cdots = \frac{4}{3}-1=\frac{1}{3}.$$

$$1+\frac{1}{4}+\frac{1}{16}+\cdots = \sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n = \frac{1}{1-1/4} = \frac{1}{3/4} = \frac{4}{3}$$

# Proof Without Words: The Alternating Harmonic Series Sums to ln 2

This was a copyrighted image, so I can't post it as part of the notes, but I'll add a link to this proof on the course webpage.

# Final Wrapup

<u>Thm</u> $\sum a_n$ converges $\Rightarrow$ $a_n \to 0$

⚠ CONVERSE $\left( a_n \to 0 \Rightarrow \sum a_n \text{ converges} \right)$
is <u>NOT</u> true

<u>Ex</u> $\sum \frac{1}{n} = +\infty$ yet $\frac{1}{n} \to 0$.

<u>Sketch of Pf</u> Intuitively, suppose $a_n \to L \neq 0$.

$$\sum a_n \approx a_1 + a_2 + a_3 + \cdots + L + L + L + L + L + \cdots \to \pm\infty$$