§8.2 Convergence Tests

Goal: Can we analyze a_n's instead of $s_n = a_1 + a_2 + \cdots + a_n$ to determine if $\sum a_n$ converges.

Theorem names in this section might sound familiar...

- comparison test
- ratio test
- root test
- integral test
- alternating series test

Basic Test for Divergence

Thm $\sum a_n$ converges $\Rightarrow a_n \to 0$

CP: $a_n \not\to 0 \Rightarrow \sum a_n$ diverges.

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{(n+1)!}{(n-1)!} = \sum_{n=1}^{\infty} \frac{(n+1)(n)\cancel{(n-1)}\cancel{(n-2)}\cdots\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{\cancel{(n-1)}\cancel{(n-2)}\cdots\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}$

$\displaystyle = \sum_{n=1}^{\infty}(n+1)n = \sum n^2+n$

↳ does not $\to 0$

$\Rightarrow$ series diverges.

⚠ worth checking b/c it's fast but most $\sum a_n$ we'll look at have $a_n \to 0$.

Thm 8.2.1 (Comparison Test) Suppose $0 \le a_n \le b_n \ \forall n$.

(1) $\sum b_n$ converges $\Rightarrow$ $\sum a_n$ converges

(2) $\sum a_n = +\infty$ $\Rightarrow$ $\sum b_n = +\infty$

⚠ book changes roles of a_n, b_n.

Pf Let $\left.\begin{array}{l} s_n = a_1 + a_2 + \cdots + a_n \\ t_n = b_1 + b_2 + \cdots + b_n \end{array}\right\}$ $s_n \le t_n$ since $a_n \le b_n \ \forall n$.

Pf of (1) $a_n \ge 0 \Rightarrow s_n$ incr'g, and thus bdd below by s_1.

By earlier sections, $s_n \le t_n \Rightarrow \lim s_n \le \lim t_n = t$

exists – its $\sum b_n$

Hence s_n bdd above by t

$\Rightarrow$ By MCT s_n converges ($\Rightarrow$ by def$^{\underline{n}}$ $\sum a_n$ too)

Keys to a Succesful Use of Comparison Test

(a) Compare to known series.

(b) make sure direction of comparison is useful.

Ex $\sum \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \dots$

Ex $0 \le \frac{1}{n^2} \le \frac{1}{n} \quad \forall n$ so

$$0 \le \sum \frac{1}{n^2} \le \sum \frac{1}{n} = +\infty$$

So comp. test tells us nothing about $\sum 1/n^2$ here

Ex $\sum \frac{1}{n(n+1)}$ (telescoping, converges to 1.)

$$\forall n \quad 0 \le \frac{1}{n(n+1)} \le \frac{1}{n^2} \Rightarrow 1 \le \sum \frac{1}{n^2}$$

$\exists$ useful comparisons!

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)^2} = \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$ (if this converges, then $\sum \frac{1}{n^2}$ converges, to $\sum \frac{1}{(n+1)^2} + 1$.)

$$0 \le \frac{1}{(n+1)(n+1)} \le \frac{1}{n(n+1)} \quad \forall n$$

$$\Rightarrow \sum \frac{1}{(n+1)^2} \le 1 \text{ and hence converges.}$$

Thm If $\sum |a_n|$ converges, so does $\sum a_n$.

Notes (1) If $\sum |a_n|$ converges we say $\sum a_n$ converges absolutely

(2) If $\sum a_n$ converges but $\sum |a_n|$ diverges, then $\sum a_n$ converges conditionally.

Ex $\sum \frac{(-1)^{n+1}}{n}$ converges, $\sum \frac{1}{n}$ diverges.