§8.3 Power Series

So far our series have been infinite sum of **preselected** numbers. Given $(a_n) = (a_1, a_2, a_3, \ldots)$ we analyze

$$\sum a_n = a_1 + a_2 + a_3 + a_4 + \cdots$$

In this section, our series are **functions** which depend on a variable. Two issues:

1. When does make sense?

2. Why would we care?

Def Let a_n be a sequence. Then $\sum_{n=0}^{\infty} a_n x^n = a_0 x^0 + a_1 x + a_2 x^2 + \cdots$ is a power series. a_n is coeff of x^n. (the n^{th} coeff)

Notes ① For a specific x, we get a regular old series

$$\sum_{n=0}^{\infty} \frac{1}{n+1} x^n = 1 + \frac{1}{2}x + \frac{1}{3}x^2 + \frac{1}{4}x^3 + \cdots$$

$\underline{x=1}$ $\quad \sum_{n=0}^{\infty} \frac{1}{n+1} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots = +\infty$

$\underline{x=\frac{1}{2}}$ $\quad \sum_{n=0}^{\infty} \frac{1}{n+1}\left(\frac{1}{2}\right)^n = 1 + \frac{1}{4} + \frac{1}{12} + \frac{1}{32} + \cdots$

Converges by... comparison test? $\frac{1}{n+1}\left(\frac{1}{2}\right)^n \leq \left(\frac{1}{2}\right)^n$

Main goal of this section: Simultaneously find $\underline{\text{all}}$ values of x for which $\sum a_n x^n$ converges.

② **WHY** do all of this?

It gives us another way to represent functions.

Ex If $x \in (-1,1)$ [so $|x|<1$]

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1+x+x^2+x^3+x^4+\dots$$

nicer than

Ex $\forall x \in \mathbb{R}$, it turns out $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\dots$

still nicer than

$\exists$ advantages to power series form. (Sometimes.)

Ex $e^u = 1 + u + \frac{u^2}{2} + \frac{u^3}{6} + \frac{u^4}{24} + \cdots$

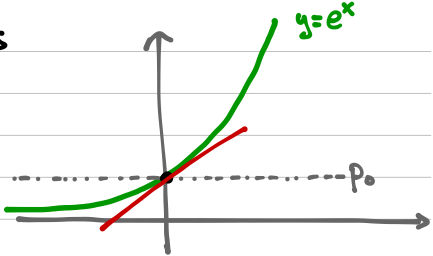
Set $u = x^2$ to get

$$e^{x^2} = 1 + (x^2) + \frac{(x^2)^2}{2} + \cdots = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \frac{x^8}{24} + \cdots$$

⚠ $\int e^{x^2}\,dx$ cannot be written using "elementary" fns, but in Advance Calc / Real Analysis we <u>prove</u> you can integrate a power series term by term:

$$\int e^{x^2}\,dx = \int \left(1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \cdots\right)dx = \left(x + \frac{x^3}{3} + \frac{x^5}{10} + \frac{x^7}{42} + \cdots\right) + C$$

③ Another way power series arise is through **Taylor Polynomials**.



e^x is hard to compute, but polynomials are "easy", esp. for a computer...

Can we find poly's of degree n, $p_n(x) \approx e^x$ near $x=0$?
(Match fn value and as many derivatives as possible.)

$p_0(x) = 1$ $\qquad$ $p_0(0) = 1 = e^0$ ✓

$p_1(x) = 1+x$ $\qquad$ $p_1(0) = 1+0 = 1 = e^0$ ✓

$p_1'(0) = 1 = 1^{st}$ deriv of e^x @ $x=0$ ✓

$$p_2(x) = 1 + x + \frac{x^2}{2}$$

Main Goal For what values of x does $\sum a_n x^n$ converge?

Think: what's the domain?

Ex $\displaystyle\sum_{n=1}^{\infty} \underbrace{\left(\frac{2^n}{n}\right)}\, x^n = 2\cdot x + 2x^2 + \frac{8}{3}x^3 + \cdots$

For some chosen x, use ratio test:

$$\lim \left| \frac{2^{n+1}\cdot x^{n+1}}{n+1} \cdot \frac{n}{2^n\, x^n} \right| = \lim_{n\to\infty} \left| \frac{2x\cdot n}{n+1} \right| = 2|x| < 1$$

if $|x| < \frac{1}{2}$, i.e. series converges $\forall\ x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$

and if $|x| > 1/2$, then lim is >1 and series div's.

We need to check cases where limit is 1 by hand.

i.e. $2|x| = 1$, $x = 1/2$ or $x = -1/2$

$\underline{x = \frac{1}{2}}$: $\sum \left(\frac{2^n}{n}\right)\left(\frac{1}{2}\right)^n = \sum \frac{1}{n} = +\infty$ divergens

$x = -1/2$ $\sum \left(\frac{2^n}{n}\right)\left(-\frac{1}{2}\right)^n = \sum (-1)^n \cdot \frac{1}{n}$ converges (alt. harm. series)

Thus series converges iff $x \in [-1/2, 1/2)$.

<u>interval</u> of conv'nce is

<u>radius</u> of conv'nce is $r = 1/2$