The following is a *non-comprehensive* list of solutions to the computational problems on the homework. For some problems there is a sketch of a solution. On other problems the stated solution may be complete. As always, feel free to ask if you are unsure of the appropriate level of details to include in your own work. Feel free to talk to us about any of the writing intensive problems.

Please let us know if you spot any typos and we'll update things as soon as possible.

(1) Suppose A has exactly two elements and B has exactly three. How many functions are there from A to B? How many are surjective? How many are injective?

For convenience, let's say $f : \{1, 2\} \to \{a, b, c\}$. To define f, we need to determine $f(1)$ and $f(2)$. There are three choices for each, so $3 \cdot 3 = 9$ total functions.

To create an injective function, I can choose any of three values for $f(1)$, but then need to choose one of the *two* remaining different values for $f(2)$, so there are $3 \cdot 2 = 6$ injective functions.

No surjective functions are possible; with two inputs, the range of f will have at most two elements, and the codomain has three elements.

(2) How does the previous question change if the functions are now from B to A?

Now there are three inputs, with two choices for each, so $2 \cdot 2 \cdot 2 = 8$ total functions. Using the same sets for A and B as above, I could do some careful counting (as in #1) to figure out how many are surjective, but it might be easier to count those that *aren't* surjective:

$$f(a) = 1, f(b) = 1, f(c) = 1$$

$$f(a) = 2, f(b) = 2, f(c) = 2$$

These are the only non-surjective functions (are you convinced?), so there are $8 - 2 = 6$ surjective functions.

No injective functions are possible in this case. An injective function would require three elements in the codomain, and there are only two.

(3) Classify each function as injective, surjective, bijective or none of these. Ask us if you're not sure why any of these answers are correct.

(a) $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = n + 3$.

Injective, but not surjective. For all n, $f(n) \neq 1$, for example.

(b) $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = n - 5$.

Injective and surjective, hence bijective.

(c) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3 - x$.

Surjective, but not injective; $f(1) = f(0)$.

(d) $f : \mathbb{N} \to \mathbb{Z}$ defined by $f(n) = n^2 - n$.

Injective, but not surjective. There is no n for which $f(n) = 1$, for example.

(e) $f : [3, \infty) \to [5, \infty)$ defined by $f(x) = (x - 3)^2 + 5$.

Injective and surjective, hence bijective.

(f) $f : \mathbb{N} \to \mathbb{Q}$ defined by $f(n) = 1/n$.

Injective, but not surjective; there is no n for which $f(n) = 3/4$, for example.

(4) In each part, find a function $f : \mathbb{N} \to \mathbb{N}$ that has the desired properties.

(a) Surjective, but not injective

One possible answer is $f(n) = \lfloor \frac{n+1}{2} \rfloor$, where $\lfloor x \rfloor$ is the *floor* or "round down" function. So $f(1) = f(2) = 1$, $f(3) = f(4) = 2$, $f(5) = f(6) = 3$, etc.

(b) Injective, but not surjective

One possible answer is $f(n) = n + 1$.

(c) Neither surjective nor injective

One possible answer is $f(n) = 2 \cdot \lfloor \frac{n+1}{2} \rfloor$. So now

$$f(1) = f(2) = 2$$
$$f(3) = f(4) = 4$$
$$f(5) = f(6) = 6$$

and so on.

(d) Bijective.

$f(n) = n$ is boring, but works nicely!

(5) Find counterexamples to each of these statements for $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$:

(a) If f and g are surjective, then $f + g$ is surjective.

Suppose $f(x) = x$ and $g(x) = -x$. Then $f + g(x) = x - x = 0$.

(b) If f and g are surjective, then $f \cdot g$ is surjective.

The same $f(x) = x$ and $g(x) = -x$ from above work; $f \cdot g(x) = -x^2$, which is not surjective.