Math 3283W
Fall 2011
Final Exam
12/17/11
Time Limit: 120 Minutes

Name (Print): ____________________

Your TA's Name: ____________________

This exam contains 11 pages (including this cover page) and 10 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Show your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. Your work should be mathematically correct and carefully and legibly written.
- **A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit**; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- Full credit will be given only for work that is presented neatly and logically; work scattered all over the page without a clear ordering will receive very little credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	30	
2	30	
3	25	
4	20	
5	20	
6	20	
7	30	
8	20	
9	25	
10	30	
Total:	250	

1. (30 points) Short answer. No justification or calculations are necessary for your answers to this question.
   (a) (8 points) Rewrite the following statement using mathematical symbols and quantifiers where possible: *for every positive real number M there exists a positive number N such that $N < 1/M$.*

   (b) (8 points) Write the negation of the following statement: *for every $x \in (0, 1)$, either $f(x) < 2$ or $f(x) > 5$.*

   (c) (5 points) Define what it means for the rational numbers to be *dense* in $\mathbb{R}$.

   (d) (5 points) For which truth values of p and q is the implication $p \Rightarrow q$ false?

   (e) (4 points) Give an example of a series which converges conditionally but not absolutely.

2. (30 points) Short answer. Your answers to these problems can be brief and concise; formal proofs are not necessary.

   (a) (6 points) Place the following sets in increasing order of cardinality. Make sure to be clear if the cardinality of a set on your list is *strictly less than* or *equal* to the next set in your list.

   $$(0,1), \quad \{1,2,3\ldots,10\}, \quad \mathbb{Q}, \quad \emptyset, \quad \mathbb{N}, \quad \mathbb{R}$$

   (b) (8 points) Give an example of a surjection $g : \mathbb{Z} \to \mathbb{N}$ which is not injective.

   (c) (8 points) Give an example of an injection $h : \mathbb{N} \to \mathbb{Z}$ which is not surjective.

   (d) (8 points) Let $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$. Give a bijection $f : \mathbb{N}_0 \to \mathbb{N}$ to show that $\mathbb{N}_0$ and $\mathbb{N}$ are equinumerous.

3. (25 points) For the purposes of this problem, define an integer $n \in \mathbb{Z}$ to be *even* if it can be written in the form $n = 2k$ for some $k \in \mathbb{Z}$, and *odd* if it can be written in the form $n = 2k + 1$ for some $k \in \mathbb{Z}$. You may use without proof that the negation of "n is even" is "n is odd," and vice versa.

   (a) (10 points) Write a direct proof of the following statement: *if n is odd, then n^2 is odd.*

   (b) (5 points) Write a one-line justification of why a correct proof of part (a) also proves the statement: *if n^2 is even, then n is even.*

   (c) (10 points) Write a proof by contradiction of the following statement: *no odd integer is the sum of two even integers.*

4. (a) (8 points) Consider the following subset of $\mathbb{R}$:

$$S = \left\{ 1 - \frac{1}{n} \ : \ n \in \mathbb{N} \right\}$$

Give values for $\sup S$, $\inf S$, $\max S$, and $\min S$, or write "none" for any of these numbers which do not exist. (No formal proofs are necessary here.)

(b) (6 points) Prove that the sequence $(s_n) = \left(1 - \frac{1}{n}\right)$ is increasing.

(c) (6 points) Prove that (s_n) converges without finding its limit; in particular, do not use the definition of convergence or the "limit laws" in Theorem 17.1 of your textbook.

5. (a) (5 points) Give an example of a relation on $\mathbb{R}$ which is reflexive and transitive, but not symmetric. Give a brief justification to support your answer.

(b) (10 points) Consider the relation R on $\mathbb{N} \times \mathbb{N}$ given by $(a, b)R(c, d)$ if and only if $ab = cd$. Prove R is an equivalence relation.

(c) (5 points) Find the equivalence class of $(2, 3)$ with the relation from part (b).

6. (20 points) Determine whether the following series converge or diverge. You may use your knowledge of when geometric series and p-series converge or diverge without proof. Justify your answers. (10 Points each)

   (a) $\displaystyle\sum_{n=0}^{\infty} \left(\sqrt{n+1} - \sqrt{n}\right)$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+n}$

7. (30 points) Find the radius and interval of convergence for each of the following power series. (15 Points each)

   (a) $\displaystyle\sum_{n=0}^{\infty}\left(\frac{1}{n\,3^n}\right)x^n$

   (b) $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$

8. (a) (15 points) Let $(s_n) = \left(\frac{n+1}{n}\right)$. Prove $s_n \to 1$ using the definition of convergence, and not any limit theorems or other results.

(b) (5 points) Let $a_1 = 1$ and $a_{n+1} = \sqrt{6 + a_n}$. You may assume without proof that (a_n) converges. Find its limit.

9. (a) (15 points) Let A_1 and A_2 be open subsets of $\mathbb{R}$. Prove $A_1 \cup A_2$ is open.

   (b) (10 points) Let C_1 and C_2 be closed subsets of $\mathbb{R}$. Prove $C_1 \cap C_2$ is closed. You may use the result in part (a).

10. (a) (15 points) Use induction to prove $1 + r + r^2 + \cdots + r^n = \dfrac{1 - r^{n+1}}{1 - r}$ for $r \neq 1$.

(b) (15 points) Let A, B and C be subsets of a universal set U. Prove: $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.