Math 3283W
Spring 2011
Final Exam
May 13, 2011
Time Limit: 120 minutes

Name (Print): ______________________
Student ID: ______________________
Section Number: ______
Teaching Assistant: ______________________
Signature: ______________________

This exam contains 10 numbered problems. Check to see if any pages are missing. Point values are in parentheses. No books, notes, or electronic devices are allowed.

1	20 pts	
2	20 pts	
3	20 pts	
4	20 pts	
5	20 pts	
6	20 pts	
7	20 pts	
8	20 pts	
9	20 pts	
10	20 pts	
TOTAL	200 pts	

1. (20 points) (4 points each) For each of the following five statements, determine whether the statement is true or false. Circle your answer. **No justification necessary.**

(a) A conditional statement is logically equivalent to its contrapositive.

TRUE FALSE

(b) The set of rational numbers, together with the operations of addition and multiplication, is a complete ordered field.

TRUE FALSE

(c) There exists a rearrangement of the terms of the alternating harmonic series that converges to zero.

TRUE FALSE

(d) For every sequence (a_n) of nonzero numbers, we have $\limsup |a_n|^{1/n} \leq \limsup |\frac{a_{n+1}}{a_n}|$.

TRUE FALSE

(e) Every member of the set $S = \{\frac{1}{n} : n \in \mathbf{N}\}$ is an isolated point of S.

TRUE FALSE

2. (20 points) (5 points each) Statements.

   a. State the Principle of Mathematical Induction. (In your answer, let $P(n)$ denote a statement about the natural number n, and write your statement in the form "If ... and ..., then ....")

   b. State the Intermediate Value Theorem.

   c. State the Heine-Borel Theorem.

   d. State the Completeness Axiom.

3. (20 points) (5 points each) Definitions. Complete each sentence.

a. A sequence (a_n) *converges* to a if ...

b. A series $\sum_{n=1}^{\infty} a_n$ *converges* to s if ...

c. A number x is a *boundary point* of a set $S \subseteq \mathbf{R}$ if ...

d. Suppose that $S \subseteq \mathbf{R}$ is nonempty and bounded below. The real number m is the *infimum* of the set S if ...

4. (20 points) (5 points each) Calculations. No justification necessary.

a. Find the limit of the convergent, recursively-defined sequence given by $a_1 = 5$ and $a_{n+1} = \sqrt{8a_n - 3}$.

b. Find the set S of subsequential limits and find $\limsup a_n$ and $\liminf a_n$ for the sequence

$$(a_n) = \left(\frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \dots\right).$$

c. Find the sum of the convergent series

$$\sum_{n=3}^{\infty} \left(\frac{1}{3}\right)^n.$$

(Note the index of the first term of the series. Write your answer in the form $\frac{m}{n}$, where m and n are natural numbers.)

d. Write the boundary and interior of the set of rational numbers, as a subset of the real numbers.

5. (20 points) (5 points each) Examples. No justification necessary.

a. Give an example of a series $\sum_{n=1}^{\infty} a_n$ that is convergent and has terms a_n that are constant.

b. Give an example of a sequence (a_n) whose terms are all distinct and positive, and $\limsup a_n = +\infty$ and $\liminf a_n = 0$.

c. Give an example of a function $f : \mathbf{R} \to \mathbf{R}$ and a subset $S \subseteq \mathbf{R}$ with the property that $f^{-1}(f(S)) \neq S$.

d. Give an example of an infinite collection $C_1, C_2, \ldots$ of closed subsets of $\mathbb{R}$ with the property that

$$\bigcup_{n=1}^{\infty} C_n$$

is not closed.

6. (20 points) Let A and B be sets, and let $f : A \to B$ be a function. Suppose that A_1 and A_2 are subsets of A. Prove that

$$f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2).$$

7. (20 points) a. (15 points) Prove that for all n, we have

$$(2)(6)(10)(14)\cdots(4n-2) = \frac{(2n)!}{n!}.$$

b. (5 points) Let us say that the *infinite product*

$$\prod_{i=1}^{\infty} a_i$$

has value P if the sequence (s_n) of partial products

$$s_n = \prod_{i=1}^{n} a_i = a_1 \cdot a_2 \cdot \cdots \cdot a_n$$

converges to P. Find the value of the infinite product

$$\prod_{n=1}^{\infty} \frac{1}{4n-2}.$$

8. (20 points) a. (5 points) Complete the following definition: $f : \mathbf{R} \to \mathbf{R}$ is *continuous* at $x = a$ if ... (Note: this definition contains nested quantifiers. Also note that $\mathbf{R}$ is the domain of f.)

b. (5 points) Write the negation of your definition in (a).

c. (10 points) Show, directly from the definition of continuity, that the function $f(x) = x^2 - 3x$ is continuous at $x = 2$.

9. (20 points) Find the set of all real numbers x for which the following series converges:

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{x+5}{3} \right)^n .$$

Show your work.

10. (20 points) (10 points each) Determine whether each of the following series converges absolutely, converges conditionally, or diverges. Justify your answers. In the case of convergence, do not find the sum.

a.

$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{\sqrt{n^4+2}}$$

b.

$$\frac{1}{2} - \frac{1}{2 \cdot 3} + \frac{1}{2^2 \cdot 3} - \frac{1}{2^2 \cdot 3^2} + \frac{1}{2^3 \cdot 3^2} - \cdots$$