Math 3283W	Name (Print):	Key & Grading Guide
Spring 2011	Student ID:	4
Final Exam	Section Number:	
May 13, 2011	Teaching Assistant:	
Time Limit: 120 minutes	Signature:	

This exam contains 10 numbered problems. Check to see if any pages are missing. Point values are in parentheses. No books, notes, or electronic devices are allowed.

1	20 pts	
2	20 pts	
3	20 pts	
4	20 pts	
5	20 pts	
6	20 pts	
7	20 pts	
8	20 pts	
9	20 pts	
10	20 pts	
TOTAL	200 pts	

1. (20 points) (4 points each) For each of the following five statements, determine whether the statement is true or false. Circle your answer. No justification necessary.

(a) A conditional statement is logically equivalent to its contrapositive.

TRUE

**FALSE** 

(b) The set of rational numbers, together with the operations of addition and multiplication, is a complete ordered field.

TRUE

FALSE

not complete

(c) There exists a rearrangement of the terms of the alternating harmonic series that converges to zero.

TRUE

**FALSE** 

(d) For every sequence  $(a_n)$  of nonzero numbers, we have  $\limsup |a_n|^{1/n} \leq \limsup |\frac{a_{n+1}}{a_n}|$ .

TRUE

**FALSE** 

(e) Every member of the set  $S = \{\frac{1}{n} : n \in \mathbb{N}\}\$  is an isolated point of S.

TRUE

**FALSE** 

- 2. (20 points) (5 points each) Statements.
  - a. State the Principle of Mathematical Induction. (In your answer, let P(n) denote a statement about the natural number n, and write your statement in the form "If ... and ..., then ....")

If 
$$P(1)$$
 is true,  
and  $P(k) \Rightarrow P(k+1)$  for all  $k$ ,  $-2$   
then  $P(n)$  is true for all  $n$ .  $-2$ 

b. State the Intermediate Value Theorem.

If  $f: [a,b] \rightarrow \mathbb{R}$  is continuous,

and if k is any number between f(a) & f(b),

then there exists c such that a < c < b & f(c) = k.

c. State the Heine-Borel Theorem.

S⊆R is compact if and only if S is closed and bounded.

2

4 €

d. State the Completeness Axiom.

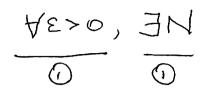
If SCR is nonempty and bounded above,

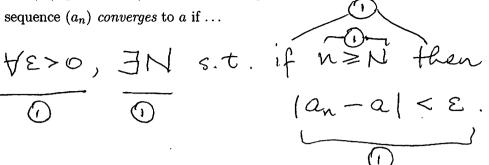
than S has a supremum.

(3)

3. (20 points) (5 points each) Definitions. Complete each sentence.

a. A sequence  $(a_n)$  converges to a if ...





b. A series  $\sum_{n=1}^{\infty} a_n$  converges to s if ...

the sequence 
$$S_n = a_1 + \dots + a_n$$
 of partial sums converges to  $S$ , as above.

c. A number x is a boundary point of a set  $S \subseteq \mathbf{R}$  if ...

$$O$$
—  $\forall \varepsilon > 0$ ,  
 $O$ —  $N(x,\varepsilon) \cap S \neq \emptyset$  and  
 $O$ —  $N(x,\varepsilon) \cap S^c \neq \emptyset$ .

d. Suppose that  $S \subseteq \mathbf{R}$  is nonempty and bounded below. The real number m is the infimum of the set S if ...

o 
$$m \in S$$
,  $\forall S \in S$ . —2  
and of  $m' \in S$ ,  $\forall S \in S$ , then  $m' \in m$ . —3  
(or contrapositive)

- 4. (20 points) (5 points each) Calculations. No justification necessary.
  - a. Find the limit of the convergent, recursively-defined sequence given by  $a_1 = 5$  and  $a_{n+1} =$  $\sqrt{8a_n-3}$ .

the limit a setisfies a = 1/8a-3. — 2

2 for solutions

for identifying

this 4-V13 is an artranomic in + has (4-V13 is an extraneous root, because the sequence increases from 5)

b. Find the set S of subsequential limits and find  $\limsup a_n$  and  $\liminf a_n$  for the sec

$$(a_n) = \left(\frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \dots\right).$$

$$S = \{ \circ \}$$
.

lim inf an = lem sup an = 0.

(1) based on S

c. Find the sum of the convergent series

$$\sum_{n=3}^{\infty} \left(\frac{1}{3}\right)^n.$$

(Note the index of the first term of the series. Write your answer in the form  $\frac{m}{n}$ , where m and

n are natural numbers.)  $\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$  (3)

 $\sum_{n=3}^{\infty} \left(\frac{1}{3}\right)^n = \frac{3}{2} - 1 - \frac{1}{3} - \frac{1}{9} = \frac{1}{18}$  (2)

d. Write the boundary and interior of the set of rational numbers, as a subset of the real numbers.

 $\partial \Omega = R$ 



5. (20 points) (5 points each) Examples. No justification necessary. All OR NOTHING a. Give an example of a series  $\sum_{n=1}^{\infty} a_n$  that is convergent and has terms  $a_n$  that are constant.

$$a_n \to 0$$
 is a necessary condition for convergence of  $\Sigma a_n$ .

The only segmence that is constant & converges to  $0$ 

is  $a_n = 0$ .

 $\sum_{n=1}^{\infty} 0$ 

b. Give an example of a sequence  $(a_n)$  whose terms are all distinct and positive, and  $\limsup a_n = +\infty$  and  $\liminf a_n = 0$ .

$$(an) = (2, \frac{1}{2}, 3, \frac{1}{3}, 4, \frac{1}{4}, \dots)$$

c. Give an example of a function  $f: \mathbf{R} \to \mathbf{R}$  and a subset  $S \subseteq \mathbf{R}$  with the property that  $f^{-1}(f(S)) \neq S$ .

$$f(x) = x^{2}$$
  
 $S = \{1\}$ .  
 $f^{-1}(f(8)) = \{-1, 1\} \neq S$ .

d. Give an example of an infinite collection  $C_1, C_2, \ldots$  of closed subsets of  $\mathbb R$  with the property that

is not closed. 
$$C_{n} = [\frac{1}{n}, 3 - \frac{1}{n}] \quad \text{closed}$$

$$C_{n} = [0, 3] \quad \text{not closed}.$$

$$n=1$$

6. (20 points) Let A and B be sets, and let  $f:A\to B$  be a function. Suppose that  $A_1$  and  $A_2$  are subsets of A. Prove that

 $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2).$ 

Pf Let 
$$y \in f(A, \cap A_2)$$
.

There exists  $x \in A, \cap A_2$  with  $f(x) = y$ .  $G(x) = y$ .  $G(x) = y$ .

 $g(x) = y \in f(A, x)$ .

7. (20 points) a. (15 points) Prove that for all n, we have

$$(2)(6)(10)(14)\cdots(4n-2) = \frac{(2n)!}{n!}.$$
When  $n=1$ , we have  $2=\frac{2!}{1!}$ .

Suppose that  $(2)(6)\cdots(4k-2) = \frac{(2k)!}{k!}$ , and  $3$ 

show the statement is true for  $n=k+1$ .

$$(2)(6)\cdots(4(k+1)-2) = (2)(6)\cdots(4k-2)(4k+2)$$

$$= \frac{(2k)!}{k!}(4k+2) \quad \text{by hypothesis}. 3$$

$$= \frac{(2k)!}{k!} \cdot 2 \cdot (2k+1) = \frac{(2k+1)!}{k!} \cdot 2$$

$$= \frac{(2k+1)!}{k!} \cdot \frac{2(k+1)}{k+1} = \frac{(2k+2)!}{(k+1)!} \quad \text{algebra}$$

$$= \frac{(2(k+1))!}{(k+1)!} \cdot \frac{2}{k!} \quad \text{form} \quad \text{Thus}, \text{ the statement is}$$
b. (5 points) Let us say that the infinite product  $\text{true for all } n$ ,
$$\lim_{k \to \infty} a_k = 0 \quad \text{induction}.$$

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has value P if the sequence  $(s_n)$  of partial products

$$s_n = \prod_{i=1}^n a_i = a_1 \cdot a_2 \cdot \dots \cdot a_n$$

converges to P. Find the value of the infinite product

By (a), we have 
$$S_n = \frac{n!}{(2n)!}$$
. (2) reason 
$$\frac{S_{n+1}}{S_n} = \frac{n+1}{(2n+1)(2n+2)} \rightarrow 0$$
. By ratio test,  $S_n \rightarrow 0$ . Thus  $TI_{4n-2} = 0$ .

8. (20 points) a. (5 points) Complete the following definition:  $f: \mathbf{R} \to \mathbf{R}$  is continuous at x = aif ... (Note: this definition contains nested quantifiers. Also note that  $\mathbf{R}$  is the domain of f.)

$$\frac{4E>0}{1}$$
,  $\frac{3S>0}{0}$  such that

if  $1x-a1< S$  then  $|f(x)-f(a)|< E$ .

b. (5 points) Write the negation of your definition in (a).

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$$\exists E > 0$$
 such that  $\forall S > 0$ ,  $\exists x$  such that

 $|x - a| < S$  AND  $|f(x) - f(a)| \ge E$ .

c. (10 points) Show, directly from the definition of continuity, that the function  $f(x) = x^2 - 3x$ is continuous at x=2.

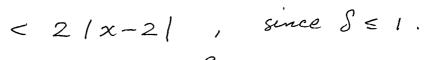
= Note that on (1,3), we have 1x-1/<2.

Let  $\varepsilon > 0$  be given. Choose  $S = \min \left\{ \frac{2}{2}, \frac{\varepsilon}{2} \right\}$ .
Then if  $1 \times -21 < S$ ,

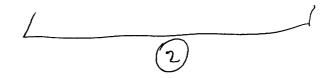
$$|f(x)-f(z)| = |x^2-3x-(-2)|$$

$$= |(x-2)(x-1)|$$

$$= |x-2|\cdot|x-1|$$
=  $|x-2|\cdot|x-1|$ 



$$<2\delta \leq 2 \cdot \frac{\varepsilon}{2} = \varepsilon.$$



9. (20 points) Find the set of all real numbers x for which the following series converges:

$$\sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{x+5}{3} \right)^n.$$

Show your work.

If 
$$b_n = \frac{1}{n} \left(\frac{x+5}{3}\right)^n$$
,  $5$  Apply ratio foot that  $\left|\frac{b_{n+1}}{b_n}\right| = \frac{n}{n+1} \cdot \frac{1}{3} |x+5|$ 

The power series converges when  $x \in [-8, -2)$ .

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Alternately:  

$$\frac{16nl''n}{3n'n} = \frac{1x+5l}{3n'n} \rightarrow \frac{1x+5l}{3} & apply root test.$$

10. (20 points) (10 points each) Determine whether each of the following series converges absolutely, converges conditionally, or diverges. Justify your answers. In the case of convergence, do not find the sum.

$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{\sqrt{n^4 + 2}}$$

$$|a_n| = \frac{n}{\sqrt{n^4 + 2}} > \frac{n}{\sqrt{n^4 + 2n^4}} = \frac{1}{n \cdot \sqrt{3}}$$

check inequality So  $\Sigma$  [an I diverges by the comparison test.]

So  $\Sigma$  [an I diverges by the comparison test.]

Thowever, since  $\frac{n}{\sqrt{n^4+2}} \to 0$  and is a decreasing function of n (no justification needed),  $\Sigma$  [an converges by the atternating series test.]

Thus Zan converges conditionally.

b. 
$$\frac{1}{2} - \frac{1}{2 \cdot 3} + \frac{1}{2^2 \cdot 3} - \frac{1}{2^2 \cdot 3^2} + \frac{1}{2^3 \cdot 3^2} - \cdots$$

$$\left| \frac{a_{n+1}}{a_n} \right| = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{3}, \text{ if } n \text{ is odd}$$

$$\left| \frac{1}{2}, \text{ if } n \text{ is even.} \right|$$

$$\lim \sup_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{2} < 1 \text{ and hence}$$

$$\sum_{n \to \infty}^{\infty} a_n \cos \beta + \sum_{n \to \infty}^{\infty} a_n \sin \beta + \sum_$$