Final Exam: Solutions

1. (a)

$$\forall M \in (0,\infty)\, \exists N \in (0,\infty),\ N < \frac{1}{M}$$

or

$$\forall M \in \mathbb{R}\, \exists N \in \mathbb{R},\ (M > 0) \wedge (N > 0) \wedge \left(N < \frac{1}{M}\right).$$

(b) The original statement is

$$\forall x \in (0,1),\ \Big(f(x) < 2\Big) \vee \Big(f(x) > 5\Big).$$

Hence, its negation takes the form

$$\exists x \in (0,1),\ \Big(f(x) \geq 2\Big) \wedge \Big(f(x) \leq 5\Big).$$

This reads: *there exists* $x \in (0,1)$ *such that* $f(x) \geq 2$ *and* $f(x) \leq 5$.

(c) (Theorem 12.12) For any two real numbers x and y such that $x < y$, there exists a rational number r satisfying $x < r < y$.

(d) The implication $p \Rightarrow q$ is false if and only if p is true, while q is false.

(e) The alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \dots$$

converges[1] but not absolutely. That's due to the fact[2] that the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

diverges.

2. (a)

$$|\varnothing| < |\{1,2,\dots,10\}| < |\mathbb{N}| = |\mathbb{Q}| < |\mathbb{R}| = |(0,1)|$$

(b) The map $g : \mathbb{Z} \to \mathbb{N}$ defined by $g(m) = |m| + 1$ for all $m \in \mathbb{Z}$ is surjective, because for any natural n, we have $n = |n-1| + 1 = g(n-1)$. On the other hand, it is not injective. Indeed, we can notice that, for instance, $g(1) = g(-1) = 2$.

(c) The map $h : \mathbb{N} \to \mathbb{Z}$ defined by $h(m) = m$ has the desired property.

(d) Such a bijection f can be constructed by setting $f(m) = m + 1$ for all $m \in \mathbb{N}_0$.

3. (a) Let n be an odd integer. Then n can be written in the form $2k+1$ for some $k \in \mathbb{Z}$. We have

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1 = 2l + 1,$$

where $l = 2k^2 + 2 \in \mathbb{Z}$. Hence, by definition of an odd integer, n^2 is odd.□

[1] See theorem 33.16.
[2] See example 32.2.

(b) The statement "*if* n^2 *is even, then* n *is even*" is the contrapositive of the statement of part (a).

(c) Suppose that there exists an odd integer n representable in the form $m_1 + m_2$, where both m_1 and m_2 are even integers. By definition of an even integer, $m_1 = 2k_1$ and $m_2 = 2k_2$ for some $k_1, k_2 \in \mathbb{Z}$. Thus, we have

$$n = m_1 + m_2 = 2k_1 + 2k_2 = 2(k_1 + k_2) = 2k,$$

where $k = k_1 + k_2 \in \mathbb{Z}$. Hence, n is an even integer. Since an integer cannot be both even and odd, we arrived to a contradiction.$\square$

4. (a)

$$\inf S = \min S = 0, \quad \sup S = 1, \quad \max S \text{ does not exist.}$$

(b) Let $m, n \in \mathbb{N}$ be such that $m < n$. Then we have

$$\frac{1}{m} > \frac{1}{n} \Rightarrow -\frac{1}{m} < -\frac{1}{n} \Rightarrow 1 - \frac{1}{m} < 1 - \frac{1}{n}.$$

The last inequality means exactly that $s_m < s_n$. Thus (s_n) is an increasing sequence. $\square$

(c) As we observed in part (a), the sequence (s_n) is bounded. In part (b) we proved that (s_n) is monotone. Therefore, by the monotone convergence theorem[3] (s_n) converges.

5. (a) The relation $\leq$ ("less than or equal to") on $\mathbb{R}$ has the desired property. Indeed, it is known to be reflexive and transitive, but it is not symmetric. For example, $0 \leq 1$ holds, but $1 \leq 0$ does not.

(b) Proving that R is an equivalence relation amounts to showing that it is reflexive, symmetric and transitive.

Reflexivity of R is immediate. Indeed, if $(a, b) \in \mathbb{N} \times \mathbb{N}$, then the trivial identity $ab = ab$ holds and, by definition of R, it means that $(a, b)R(a, b)$.

In a similar fashion one checks that R is symmetric. If $(a, b) \in \mathbb{N} \times \mathbb{N}$, then $ab = ba$. Hence, according to the definition of R, we have $(a, b)R(b, a)$.

To check transitivity, consider $(a, b), (c, d), (e, f) \in \mathbb{N} \times \mathbb{N}$ such that $(a, b)R(c, d)$ and $(c, d)R(e, f)$. Thus, we have $ab = cd$, $cd = ef$. Then $ab = ef$ and it means precisely that $(a, b)R(e, f)$. Hence, R is indeed transitive.$\square$

(c) The equivalence class of element $(2, 3)$ with respect to the relation R consists of all pairs $(a, b) \in \mathbb{N} \times \mathbb{N}$ such that $(2, 3)R(a, b)$. That is, $ab = 6$. This equation has exactly four solutions in natural numbers. Namely, $a = 1, b = 6$, $a = 2, b = 3$, $a = 3, b = 2$ and $a = 6, b = 1$. Hence,

$$E_{(2,3)} = \{(1, 6), (2, 3), (3, 2), (6, 1)\}.$$

6. (a) We can rewrite the given series as follows:

$$\begin{aligned}\sum_{n=0}^{\infty}(\sqrt{n+1} - \sqrt{n}) &= \sum_{n=0}^{\infty} \frac{(\sqrt{n+1} + \sqrt{n}) \cdot (\sqrt{n+1} - \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} \\ &= \sum_{n=0}^{\infty} \frac{(n+1) - n}{\sqrt{n+1} + \sqrt{n}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}}.\end{aligned}$$

[3] See theorem 18.3.

Now, recall that the p-series[4] $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum\limits_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$ diverges, and observe that for any $n \geq 0$, the inequality

$$\frac{1}{\sqrt{n+1}+\sqrt{n}} \geq \frac{1}{\sqrt{n+1}+\sqrt{n+1}} = \frac{1}{2} \cdot \frac{1}{\sqrt{n+1}}.$$

holds. Hence, by the comparison test, the given series diverges as well.

(b) First, notice that the p-series $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ converges ($p > 1$). Now, observe that for any $n \geq 1$, we have

$$\frac{1}{n^2+n} < \frac{1}{n^2}.$$

Hence, by the comparison test, the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^2+n}$ converges.

7. (a) First, we compute

$$\lim \frac{|a_{n+1}(x)|}{|a_n(x)|} = \lim \frac{|x^{n+1}|}{(n+1)3^{n+1}} \cdot \frac{n3^n}{|x^n|} = \lim \frac{n}{n+1} \cdot \frac{x}{3} = \frac{|x|}{3}.$$

The ratio test guarantees that the series $\sum\limits_{n=1}^{\infty} \left(\frac{1}{n3^n}\right) x^n$ converges if $\frac{|x|}{3} < 1$ or, equivalently, $|x| < 3$. Hence, the radius of convergence of the given series is 3. Now, we need to investigate the behavior of the given series at the points $x = \pm 3$. For $x = 3$, the series becomes

$$\sum_{n=1}^{\infty} \left(\frac{1}{n3^n}\right) \cdot 3^n = \sum_{n=1}^{\infty} \frac{1}{n}.$$

This is the harmonic series and it is known to diverge.
At $x = -3$ we have

$$\sum_{n=1}^{\infty} \left(\frac{1}{n3^n}\right) \cdot (-3)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

This is an alternating series with the absolute value of terms monotonically decreasing to zero. Hence, by the alternating series test, it converges.
Thus, the interval of convergence of the given power series is $[-3, 3)$.

(b) As in the previous part, we start by computing the limit of the ratio $\left|\frac{a_{n+1}(x)}{a_n(x)}\right|$:

$$\lim \frac{|a_{n+1}(x)|}{|a_n(x)|} = \lim \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} = \lim \frac{x}{n+1} = 0 < 1.$$

Thus, by the ratio test, the given series converges for any $x \in \mathbb{R}$.

8. (a) Let $\epsilon > 0$. Then for any natural $n > \frac{1}{\epsilon}$, we have

$$|s_n - 1| = \left|\frac{n+1}{n} - 1\right| = \left|\left(1 + \frac{1}{n}\right) - 1\right| = \frac{1}{n} < \epsilon.$$

Hence, by definition of limit, $\lim s_n = 1$. □

[4] Here, $p = 1/2$.

(b) Let $a = \lim a_n$.Using theorem 17.1, we compute $\lim a_{n+1}^2 = \lim a_n^2 = \lim a_n \cdot \lim a_n = a^2$. On the other hand, since $a_{n+1}^2 = 6 + a_n$, then $\lim a_{n+1}^2 = \lim(6 + a_n) = 6 + a$. Solving the quadratic equation $a^2 = 6 + a$ for a, we find that $a = 3$ or $a = -2$. Since all the terms of our sequence are non-negative[5], then $a = -2$ cannot be the limit of (a_n). Hence, $\lim a_n = 3$.

9. (a) Suppose first that at least one of the sets A_1 and A_2 is non-empty. Let $x \in A_1 \cup A_2 \neq \varnothing$. Then, by definition of union, $x \in A_i$ for $i = 1$ or $i = 2$. Since A_i is open, there exists an open neighborhood $N(x;\epsilon)$ of x contained in A_i. Hence, $N(x;\epsilon) \subseteq A_1 \cup A_2$, and it follows now that x is an interior point of $A_1 \cup A_2$. Therefore, $A_1 \cup A_2$ is open.

   If both A_1 and A_2 are empty, then $A_1 \cup A_2 = \varnothing$, and the empty set is open.□

   (b) By definition of a closed set, proving that $C_1 \cap C_2$ is closed is equivalent to showing that the complement $\mathbb{R} \setminus (C_1 \cap C_2)$ is open.

   We have[6]

$$\mathbb{R} \setminus (C_1 \cap C_2) = (\mathbb{R} \setminus C_1) \cup (\mathbb{R} \setminus C_2).$$

   Since each set C_i is closed, then the right-hand side of the last identity is actually a union of two open sets $\mathbb{R} \setminus C_i$ $(i = 1, 2)$. By the result of part (a), this union is open. Hence, $C_1 \cap C_2$ is closed.□

10. (a) To establish the basis of induction, we verify that the identity $1 + r + \cdots + r^n = \frac{1-r^{n+1}}{1-r}$, where $r \neq 1$, holds for $n = 1$. For $n = 1$, the left-hand side is equal to $1 + r$, and on the right-hand side we have $\frac{1-r^2}{1-r} = \frac{(1-r)(1+r)}{1-r} = 1 + r$.

   Assume now that the identity holds for $n = k$ and let us pass to the case $n = k+1$. We have

$$\begin{aligned} 1 + r + \cdots + r^{k+1} &= (1 + r + \cdots + r^k) + r^{k+1} = \frac{1 - r^{k+1}}{1 - r} + r^{k+1} \\ &= \frac{1 - r^{k+1}}{1 - r} + \frac{r^{k+1}(1 - r)}{1 - r} \\ &= \frac{(1 - r^{k+1}) + (r^{k+1} - r^{k+2})}{1 - r} = \frac{1 - r^{k+2}}{1 - r}. \end{aligned}$$

   This proves the induction step.□

   (b) Proving that the set $A \setminus (B \cup C)$ equals to $(A \setminus B) \cap (A \setminus C)$ is equivalent to showing that $A \setminus (B \cup C) \subseteq (A \setminus B) \cap (A \setminus C)$ and $A \setminus (B \cup C) \supseteq (A \setminus B) \cap (A \setminus C)$.

   "$\subseteq$" Let $x \in A \setminus (B \cup C)$. Then, by definition of complement, $x \in A$ and $x \notin B \cup C$. In particular, $x \notin B$ and $x \notin C$. Hence, $x \in A \setminus B$ and $x \in A \setminus C$. Therefore, by definition of intersection, $x \in (A \setminus B) \cap (A \setminus C)$.

   "$\supseteq$" Let $x \in (A \setminus B) \cap (A \setminus C)$. Then $x \in A \setminus B$ and $x \in A \setminus C$. It means that $x \in A$, but $x \notin B$ and $x \notin C$. Hence $x \notin B \cup C$. Therefore, $x \in A \setminus (B \cup C)$.□

[5]That's because $\sqrt{t} \geq 0$ for any real $t \geq 0$.

[6]Use De Morgan's laws or theorem 5.13(g).