The following is a non-comprehensive list of solutions to homework problems. In some cases I may give an answer with just a few words of explanation. On other problems the stated solution may be complete. As always, feel free to ask if you are unsure of the appropriate level of details to include in your own work.

Please let me know if you spot any typos and I'll update things as soon as possible.

1.5: This problem can be done with a bit of relabeling. Start with $\{P + t(Q - P)\}$, which represents the lne $\overleftrightarrow{PQ}$. (We discussed it in class, but if you don't remember why $\overleftrightarrow{PQ} = \{P + t(Q - P) \mid t \in \mathbb{R}\}$, you should talk to me soon, because we'll use that equation over and over this semester.)

$$\begin{aligned}\overleftrightarrow{PQ} &= \{P + t(Q - P) \mid t \in \mathbb{R}\}\\ &= \{P + tQ - tP \mid t \in \mathbb{R}\}\\ &= \{(1 - t)P + tQ \mid t \in \mathbb{R}\}\\ &= \{aP + bQ \mid a, b \in \mathbb{R} \text{ and } a + b = 1\}\end{aligned}$$

Where the last line just comes from relabeling $(1 - t)$ and t as a and b, respectively, which means $a + b = 1 - t + t = 1$.

Alternatively, you could start with $a+b = 1$, solve for one of the variables, say $a = 1-b$,- and substitute:

$$\begin{aligned}aP + bQ &= (1 - b)P + bQ\\ &= P + bQ - bP\\ &= P + b(P - Q)\end{aligned}$$

Because b can take on all real values, this equation gives the entire line $\overleftrightarrow{PQ}$.

1.28: Recall from class that a parametric equation for a line has the form $P + sU$; *special* means that the direction vector U is a unit vector, $||U|| = 1$. Similarly, a normal equation of a line, $\langle A, X - Y\rangle = 0$ (or $\langle A, X\rangle = c$) is *special* if the coefficient vector A is a unit vector.

I've listed answers below; talk to me if it's not clear to you why they're correct. Remember that there are many possible correct answers to each question. (And many possible incorrect answers too...)

(i) $(3, 4) + s(1, 2)$. This line contains the point $(3, 4)$ and goes in the direction of $(1, 2)$. To create the special parametric form, we need to "normalize" the vector $(1, 2)$, which is a fancy term for making it a unit vector:

$$\frac{(1, 2)}{||(1, 2)||} = \frac{1, 2}{\sqrt{1 + 4}} = \frac{(1, 2)}{\sqrt{5}} = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$$

So a special parametric form is:

$$\{(3, 4) + s(1/\sqrt{5}, 2/\sqrt{5})\}.$$

The vector $(-2/\sqrt{5}, 1/\sqrt{5})$ is perpendicular to the line and unit length. (Think about why these facts are true!) Hence a special normal form is:

$$\{\langle(-2/\sqrt{5}, 1/\sqrt{5}), X - (3,4)\rangle = 0\}$$

(iii) $t(-7, 24)$. (Let $t = 0$ to see that this line contains the origin $(0,0)$.) You can check that $||(-7, 24)|| = 25$. Special parametric and normal forms are

$$\{(0,0) + t(-7/25, 24/25) \,|\, t \in \mathbb{R}\}$$
$$\{\langle(24, 25, 7/25), X - (0,0)\rangle = 0\}$$

(v) $\langle(-3,-5), X\rangle = \sqrt{68}$ You can check that $(-3,-5)$ has length $\sqrt{34}$. We need to know a point on the line. One way to find one is to guess that there might be a point $X = (x_1, x_2)$ on the line for which $x_1 = 0$.[1] So plug $X = (0, x_2)$ into the equation and solve for $x_2 = -\sqrt{68}/5$. Hence the point $(0, -\sqrt{68}/5)$ is on the line. Special parametric and normal forms are:

$$\{(0, -\sqrt{68}/5) + t(5/\sqrt{34}, -3/\sqrt{34}) \,|\, t \in \mathbb{R}\}$$
$$\{\langle(-3/\sqrt{34}, -5/\sqrt{34}), X - (0, -\sqrt{68}/5)\rangle = 0\}$$

1.28: We defined lines to be parallel if they have direction vectors which are multiples of each other. So if $l = \{Q + sU\}$, then it's certainly true that the line $k = \{P + sU\}$ is parallel to l (since it has the same direction indicator) and is incident with P.

It remains to prove that this is the *unique* such line. So let

$$m = \{P + sV\}$$

be any line through P which is parallel to l. By definition of parallel lines, that means V (m's direction indicator) must be a multiple of U (l's direction indicator),

$$V = cU, \text{ some } c \in \mathbb{R}.$$

But U is also the direction indicator of k, meaning m has a direction indicator which is a multiple of k's. Hence $m||k$ and, since they share the point P, they must be the same line.

If you wish, you can use the third part of Proposition 1.6 to prove the lines are identical.

1.35: This problem is an extension of #5, so we can re-use our work from there:

$$P + s(Q - P) = aP + bQ,$$

with $a = (1 - s)$ and $b = s$, so $a + b = 1$. The only difference here is that for a line $\overleftrightarrow{PQ}$ we let s be any real number, but for the ray $\overrightarrow{PQ}$, $s \geq 0$. For the line segment $\bar{PQ}$ the range on s is even more restrictive, $0 \leq s \leq 1$. (We discussed this in class and you can find it in section 1.4 of the textbook.) What remains is to discuss how these restrictions on s affect the possible values of a and b:

- For the ray we want $s > 0$. Since $b = s$, this means $b > 0$, and we still have $a + b = 1$.
- For the line segment we want $0 \leq s \leq 1$. Hence $0 \leq b \leq 1$. Because $a = 1 - s = 1 - b$, that means a is also between 0 and 1, and $a + b = 1$.

[1] In other words, we're looking for the 'y-intercept' of the line, which will always exist unless a line is vertical, like $x = 2$.

1.41: Here is one possible approach, which avoids any calculations with the components of U and V. First, assume $\langle U, V\rangle = 1$. To prove $U = V$ it suffices to show $||U - V|| = 0$ or, equivalently, $||U - V||^2 = 0$. Using Lemma 1.48, i.e. the "Algebraic Law of Cosines,"

$$||U - V||^2 = ||U||^2 + ||V||^2 - 2\langle U, V\rangle = 1 + 1 - 2(1) = 0$$

To prove the second assertion when $\langle U, V\rangle = -1$ you would use the alternate equation in Lemma 1.48:

$$||U + V||^2 = ||U||^2 + ||V||^2 + 2\langle U, V\rangle = 1 + 1 + 2(-1) = 0$$

Hence $||U + V|| = 0$ or, equivalently, $U + V = 0$ and $U = -V$.

1.45: As with the previous problem, it is easier to work with distance *squared*. So we'll show the equivalent statement $||U + \alpha V||^2 = ||V + \alpha U||^2$ for unit vectors U and V.

$$\begin{aligned} ||U + \alpha V||^2 &= \langle U + \alpha V, U + \alpha V\rangle \\ &= \langle U, U\rangle + 2\alpha\langle U, V\rangle + \alpha^2\langle V, V\rangle \\ &= ||U||^2 + 2\alpha\langle U, V\rangle + \alpha^2||V||^2 \\ &= 1 + 2\alpha\langle U, V\rangle + \alpha^2 \end{aligned}$$

If you do similar work with $||V + \alpha U||^2$ you'll get the same expression.

STYLE NOTE. You should avoid "two-sided" proofs where you set the expressions equal to each other and work on both sides at once, before deciding that they are equal. In other words, don't use the following "proof."

$$\begin{aligned} ||U + \alpha V||^2 &= ||V + \alpha U||^2 \\ \langle U + \alpha V, U + \alpha V\rangle &= \langle V + \alpha U, V + \alpha U\rangle \\ \langle U, U\rangle + 2\alpha\langle U, V\rangle + \alpha^2\langle V, V\rangle &= \langle V, V\rangle + 2\alpha\langle U, V\rangle + \alpha^2\langle U, U\rangle \\ ||U||^2 + 2\alpha\langle U, V\rangle + \alpha^2||V||^2 &= ||V||^2 + 2\alpha\langle U, V\rangle + \alpha^2||U||^2 \\ 1 + 2\alpha\langle U, V\rangle + \alpha^2 &= 1 + 2\alpha\langle U, V\rangle + \alpha^2 \\ 1 &= 1 \end{aligned}$$

You've probably been told in the past that it's best to avoid this style of work – with trig identities, certain proofs by induction, and so on. Why don't we like this style? Any number of reasons. Stylistically, some will object that we're setting things equal without knowing if they're actually equal. You could avoid this by using $\stackrel{?}{=}$ instead of $=$, but that's not much better. There are mathematical issues here as well. In particular, if you work on both sides at once, you're liable to accidentally do something (like multiple both sides by 0) which could take a false equality and make it true, or vice versa. A famous example would be this "proof" that $0 = 1$: https://www.math.hmc.edu/funfacts/ffiles/10001.1-8.shtml