The following is a non-comprehensive list of solutions to homework problems. In some cases I may give an answer with just a few words of explanation. On other problems the stated solution may be complete. As always, feel free to ask if you are unsure of the appropriate level of details to include in your own work.

Please let me know if you spot any typos and I'll update things as soon as possible.

2.8: Let $P = (3, 2)$, $Q = (2, 0)$ and $R = (4, 5)$. Here are the steps to measure the angle $\angle PQR$. You might not have shown this much detail, which is fine, but it's vital that you find *unit* direction indicators for the angle.

(1) Find a unit vector U which is a direction indicator for $\overrightarrow{QP}$:

$$U = \frac{P - Q}{||P - Q||} = \frac{(1, 2)}{\sqrt{1 + 4}} = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right)$$

.

(2) Find a unit vector V which is a direction indicator for $\overrightarrow{QR}$:

$$V = \frac{R - Q}{||R - Q||} = \frac{(2, 5)}{\sqrt{4 + 25}} = \left(\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \right)$$
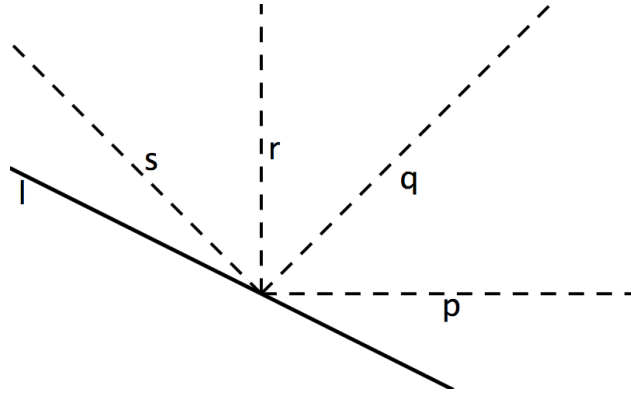
.

(3) Find the dot product of U and V:

$$\langle U, V \rangle = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right) \cdot \left(\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \right)$$
$$= \left(\frac{2}{\sqrt{145}} + \frac{10}{\sqrt{145}} \right)$$
$$= \frac{12}{\sqrt{145}}$$

(4) Find the measure of $\angle PQR$:

$$|\angle PQR| = \int_{\langle U, V \rangle}^{1} \frac{dt}{\sqrt{1 - t^2}} = \int_{\frac{12}{\sqrt{145}}}^{1} \frac{dt}{\sqrt{1 - t^2}} = 0$$

(Wow, that's ugly!)

2.17: The situation looks something like this:

We can assume Proposition 2.21, but have to be careful about only applying it to two angles at once, each of which must meet along a common ray in the interior of the outer rays. Then

$$|\angle(p,s)| = |\angle(p,q)| + (|\angle(q,s)|)$$
$$= |\angle(p,q)| + (|\angle(q,r)| + |\angle(r,s)|)$$

2.18: There are many possible answers. Here's one. Let p be the positive x-axis, q the positive y-axis, r the negative x-axis and s the negative y-axis:
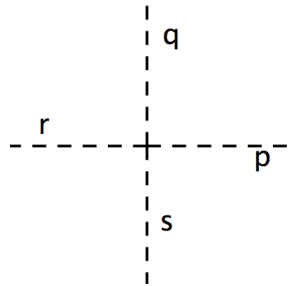
With our definitions, $|\angle(p,s)| = \pi/2$, but:

$$|\angle(p,q)| + |\angle(q,r)| + |\angle(r,s)| = \pi/2 + \pi2 + \pi2 = 3\pi/2.$$

2.24: First, we defined $\sin x$ to be $\sqrt{1 - \cos^2 x} = \left(1 - \cos^2 x\right)^{\frac{1}{2}}$, and we found $\frac{d}{dx}\cos x = -\sin x$. But you don't yet know the derivative of $\sin x$, unless you compute it as part of this problem. Notice this only uses the chain rule and the derivative of $\cos x$:

$$\frac{d}{dx}\sin x = \frac{1}{2}(1 - \cos^2 x)^{-\frac{1}{2}}(-2\cos x)(\cos x)' = \frac{2\cos x \sin x}{2\sqrt{1 - \cos^2 x}} = \frac{2\cos x \sin x}{2\sin x} = \cos x$$

2

Now we can use the quotient rule to find the derivative of $\tan x$.

$$\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x}$$
$$= \frac{(\cos x)(\sin x)' - (\sin x)(\cos x)'}{\cos^2 x}$$
$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$
$$= \frac{1}{\cos^2 x}$$

Because its derivative is always positive on its domain, i.e. all x for which $\cos x \neq 0$, which amounts to $\{x \neq \frac{\pi}{2} + k\pi : k \text{ any integer}\}$, the tangent function is always increasing. So if you look at one particular period, say $(-\pi/2, \pi/2)$, then the tangent function is injective. (If you look at more than one period at a time, it's no longer one-to-one. Why?)

2.35: We did this problem as group work in class, although it was phrased a little bit differently; on that sheet I gave you vectors whose dot product was $1/2$ and essentially asked you to find $\arccos\langle U, V\rangle$. This problem asks for $\arccos 1/2$, leaving you to find the vectors. Since a number of people were gone that day, I'll include a longer discussion about the problem here.

I assigned this problem because it helps demonstrate a quirk of our definition of angle measure. Go back and look at the solution for 2.8 to refresh your memory: to find the measure of an angle, we need two *unit* vectors U and V which define the angle, and then the measure is defined as an integral:

$$\int_{\langle U,V\rangle}^{1} \frac{dt}{\sqrt{1-t^2}}$$

That definition is great mathematically, but it's a problem from a practical point of view since we can't actually evaluate that integral! So we have to resort to sneakiness:

(1) We *define* π to be the measure of an angle for which $\langle U, V\rangle = -1$. As it happens, any such U and V represent a straight angle, as we would expect. [You actually proved on the first homework assignment that $U = -V$ in this case.]

(2) In class, we asked what would happen if $\langle U, V\rangle = 0$. By comparing the resulting integral to the definition of π, we decided such an angle would have measure $\pi/2$. So that's why a right angle has measure $\pi/2$.

(3) On the worksheet (or Example 2.29 in the book), we chose U, V and W which happen to construct *two* angles which must combine to form one right angle. (Draw a picture of these vectors!) We know those angles have equal measure, since $\langle U, V\rangle = \langle V, W\rangle$, and the measure of an angle is entirely determined by the dot product of its (unit) direction indicators. So if two equal angles add to $\pi/2$, they must each measure $\pi/4$.

Following this pattern, you might try to pick U, V and W which form two angles which add up to $\pi/2$ such that $|\angle UOV|$ is twice as large as $|\angle VOW|$. Then one would have to be

$\pi/3$ and the other $\pi/6$. Using our prior knowledge of calculus and 30-60-90 triangles, you can even say what V would be:

$$U = (1, 0)$$
$$V = (1/2, \sqrt{3}/2)$$
$$W = (0, 1)$$

The problem is that its very hard to show that one of those angles is twice the other, i.e.:

$$\int_{\langle U,V \rangle}^{1} \frac{dt}{\sqrt{1 - t^2}} = 2 \int_{\langle V,W \rangle}^{1} \frac{dt}{\sqrt{1 - t^2}}$$
$$\int_{1/2}^{1} \frac{dt}{\sqrt{1 - t^2}} = 2 \int_{\sqrt{3}/2}^{1} \frac{dt}{\sqrt{1 - t^2}}$$

It turns out to be easier to choose:

$$U = (1, 0)$$
$$V = (1/2, \sqrt{3}/2)$$
$$W = (-1/2, \sqrt{3}/2)$$
$$X = (-1, 0)$$

You can verify that $\langle U, V \rangle$, $\langle V, W \rangle$ and $\langle W, X \rangle$ are all equal, so the angles have equal measure. All three of them must add up to π by our previous work. (Problem 2.17, for example – or this specific situation is stated in Problem 2.19, which follows immediately from 2.17.) Hence each is one third of π – also known as $\pi/3$. Since $\dot{U,}V = 1/2$, we have $|\angle UOV| = \arccos 1/2 = \pi/3$. Then you can combine angles. $|\angle UOW|$ must be $2\pi/3 = 2\pi/3$, and $\langle U, W \rangle = -1/2$, which mean $\arccos(-1/2) = 2\pi/3$.

2.36: Let U, V, and W span rays p, q and r, respectively, all from the same point. If $U = -W$, then you can apply Proposition 2.23 directly. Otherwise, U and W form a (non-straight) angle, and $V = aU + bW$ with $a, b > 0$ ensures that V points in the interior, so q is between p and r. and you can use Proposition 2.2

Remember that $|\angle(p, r)| = \arccos \langle U, W \rangle = \int_{\langle U,W \rangle}^{1} \frac{1}{\sqrt{1 - t^2}} dt$ are essentially synonyms!