The following is a non-comprehensive list of solutions to homework problems. In some cases I may give an answer with just a few words of explanation. On other problems the stated solution may be complete. As always, feel free to ask if you are unsure of the appropriate level of details to include in your own work.

Please let me know if you spot any typos and I'll update things as soon as possible.

3.57: Following the hint from the email I sent out, define the following functions:

$$\begin{aligned} \mathcal{M}_1(X) &= \text{ Reflection across } x_2 = 4 \\ \mathcal{T}_1(X) &= \text{ Translation by } (-5, 0) \\ \mathcal{M}_2(X) &= \text{ Reflection across } x_2 = -4 \\ \mathcal{T}_2(X) &= \text{ Translation by } (5, 0) \end{aligned}$$

Then the two glide reflections are $\mathcal{G}_1(X) = \mathcal{M}_1 \circ \mathcal{T}_1(X) = \mathcal{T}_1 \circ \mathcal{M}_1(X)$ and $\mathcal{G}_2(X) = \mathcal{M}_2 \circ \mathcal{T}_2(X) = \mathcal{T}_2 \circ \mathcal{M}_2(X)$. (Make sure you realize why I can write the compositions within each glide reflection in either order; talk to me if you have questions.) The other important observation from the email is that $\mathcal{T}_1$ and $\mathcal{T}_2$ are inverses of each other, so I can write out the composition of the glide reflections in a way that makes them cancel:

$$\begin{aligned} (\mathcal{M}_1 \circ \mathcal{T}_1) \circ (\mathcal{T}_2 \circ \mathcal{M}_2) (X) &= \mathcal{M}_1 \circ (\mathcal{T}_1 \circ \mathcal{T}_2) \circ \mathcal{M}_2(X) \\ &= \mathcal{M}_1 \circ \mathcal{M}_2(X) \end{aligned}$$

So the overall effect of composing these glide reflections is the same as reflection across $x_2 = -4$ followed by reflection across $x_2 = 4$. Those lines are parallel, so we know that's a translation, by twice the vector from the first line to the second. Hence the final answer is the translation

$$X \to X + 2 \begin{bmatrix} 0 \\ 8 \end{bmatrix} = X + \begin{bmatrix} 0 \\ 16 \end{bmatrix}$$

4.2, 4.3: The solution to 4.2 is

$$(1, 2, -2)^\triangle = 1 \cdot (3, -2) + 2(4, -2) - 2(4, -6) = (3, 6)$$

For 4.3 we have to set up a system of equations. As discussed in class and email, if you setup the correct system, there's no need in this 5000 level math course to show all of the work to find the solution to that system. Remember to include $r + s + t = 1$ in each system! For $(0, 0)$ we have:

$$\begin{aligned} 3r + 4s + 4t &= 0 \\ -2r - 2s - 6t &= 0 \\ r + s + t &= 1 \end{aligned}$$

which has solution $(r, s, t) = (4, -5/2, -1/2)$, meaning $(0, 0) = (4, -5/2, -1/2)^\triangle$. With the other system, we find that $(4, 5) = (0, 11/4, -7/4)^\triangle$.

5.4: The triangles for which the orthocenter is also a vertex are *right* triangles. For full credit you should explain both direction: why the orthocenter of a right triangle is a vertex, and why that's not the case for other triangles. (Another way to state the second part is to show that if the orthocenter is a vertex, then it can only be a right triangle. Alternatively, it's possible that your explanation for why the orthocenter of a right triangle is a vertex is actually an if-and-only-if proof.)

Start with a right triangle $\triangle ABC$ with right angle at C. Then $\overleftrightarrow{AC}$ is the altitude from A, and $\overleftrightarrow{BC}$ is the altitude from B. (Draw a picture and convince yourself of this!) Since two altitudes intersect at C, all three will intersect there, and C is the orthocenter.

Conversely, suppose one of the vertices – let's say C – is the orthocenter of a triangle. Then the altitude from A goes through C, which means the altitude actually is $\overline{AC}$, and by definition of altitude, $\overline{AC} \perp \overline{BC}$. If those two sides are perpendicular, it's a right triangle.

5.6: You could do this geometrically or with barycentric coordinates; as discussed via email, I think barycentric coordinates is easier. If $\triangle ABC$ is equilateral, then $a = b = c$, so the incenter (by Proposition 5.19) is

$$\left(\frac{a}{a+b+c}, \frac{b}{a+b+c}, \frac{c}{a+b+c}\right)^{\triangle} = \left(\frac{a}{a+a+a}, \frac{a}{a+a+a}, \frac{a}{a+a+a}\right)^{\triangle} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)^{\triangle},$$

which is the centroid. Conversely, suppose the centroid and incenter are equal:

$$\left(\frac{a}{a+b+c}, \frac{b}{a+b+c}, \frac{a}{a+b+c}\right)^{\triangle} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)^{\triangle}$$

Equating the parts of the barycentric coordinates (and multiplying by 3) gives three equations:
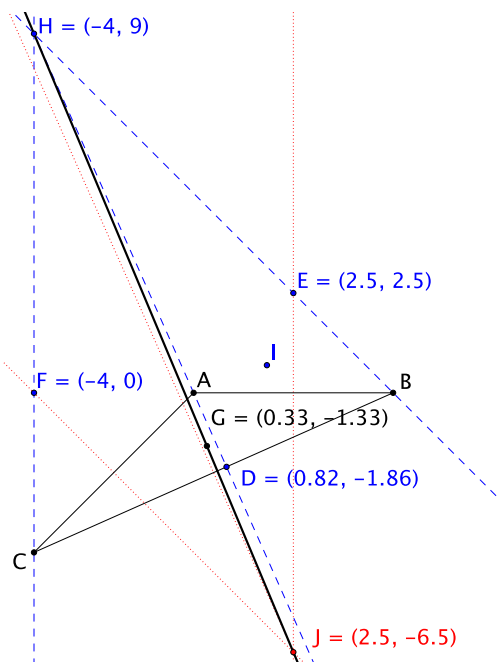
$$3a = a + b + c$$
$$3b = a + b + c$$
$$3c = a + b + c$$

You can check that this system of equations is satifsfied if and only if $a = b = c$, so that the triangle is equilateral.

5.13: I mentioned in office hours and via email that you should show more work than a GeoGebra picture for this problem, but don't have to write out systems of equations and their solutions—once you demonstrate that you know the equations of perpendicular bisectors, altitudes, or whatever, then you can use a calculator, wolfram alpha, or another tool (even GeoGebra!) to find their intersections. I've included a GeoGebra picture below for you to check your answers. I've labeled the vertices of the triangle A, B and C. The blue dashed lines are the altitudes, with feet D, E and F and orthocenter H. The red dotted lines are the perpendicular bisectors with circumcenter J. The circumradius is the distance from J to any of the vertices. (They're all the same distance from J; that's the whole point!)

$$||J - A|| = \sqrt{(5/2 - 0)^2 + (13/2 - 0)^2} = \sqrt{97/2}$$

The Euler line is the dark black line through H, J and the centroid G. You can use techniques from Chapter 1 to write down an equation of it – for example,

$$aH + bG = a(-4, 9) + b(1/3, -4/3), \qquad a + b = 1$$

$H = (-4, 9)$

$E = (2.5, 2.5)$

$F = (-4, 0)$ A I B

$G = (0.33, -1.33)$

$D = (0.82, -1.86)$

C

$J = (2.5, -6.5)$

5.20: It's possible to set up huge systems of equations with the barycentric coordinates of the incenter and orthocenter for this problem, and show they're equal if and only if $a = b = c$, but that would be... messy. I mentioned via email that a geometric approach seems simplest to me for this problem. If you had trouble solving this, use the diagram below (draw a fresh copy for each direction!) and fill in the details as you read through the proof. The letters a, b, c, α, β, and γ have their normal meanings for triangle $\triangle ABC$.

The crucial observation, as mentioned in my email, is that the incenter and orthocenter are equal if and only if each angle bisector is also an altitude.
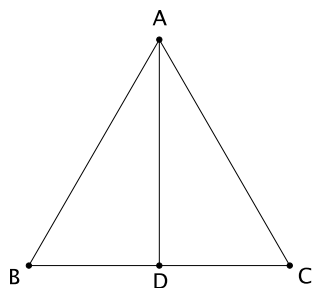
Suppose $\triangle ABC$ is equilateral. We need to show an angle bisector is also an altitude, so draw the bisector of $\angle BAC$, and label its intersection with $\overline{BC}$ as point D. We have $\triangle BAD \cong \triangle CAD$ by SAS: $\overline{BA} \cong \overline{CA}$, $\angle BAD \cong \angle CAD$, and both triangles share the side $\overline{AD}$. By CPCTC, angles $\angle ADB$ and $\angle ADC$ are congruent and add to π. Hence they are right angles, and $\overline{AD}$ is an altitude. We could repeat the same work for the other angle bisectors and altitudes in the triangle.

Conversely, suppose we know the angle bisectors are also altitudes, so starting with a fresh copy of our diagram above, the line $\overline{AD}$ is an angle bisector and an altitude. This time the triangles $\triangle ADB$ and $\triangle ADC$ are congruent by ASA, and CPCTC tells us $\overline{AB} \cong \overline{AC}$. If we repeated this with the angle bisector and altitude from C, we'd get $\overline{AC} \cong \overline{BC}$. Hence

$$\overline{AB} \cong \overline{AC} \cong \overline{BC}$$

and the triangle is equilateral.