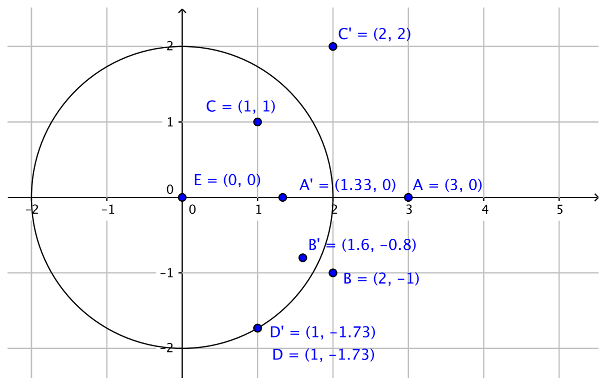
The following is a non-comprehensive list of solutions to homework problems. In some cases I may give an answer with just a few words of explanation. On other problems the stated solution may be complete. As always, feel free to ask if you are unsure of the appropriate level of details to include in your own work. Please let me know if you spot any typos and I'll update things as soon as possible.

Update: The original version of this document had the solution to 7.3, not 7.4, sorry – some of the problem numbers changed from edition to edition and I didn't catch that when writing up solutions. I left the solutions from 7.3 here in case you'd like some extra practice.

7.3: The diagram below shows each of these points and their inversions across the circle. The point $E = (0, 0)$ is inverted to $E' = \infty$, which is not shown on the picture.



7.4: You can almost use the formula given in the book for a circle inversion, with one exception: that formula assumes the mirror is centered at the origin, whereas our circle is centered at $(-8, 13)$. So we need to first move everything so the center is at the origin, then invert, and then move it back:

$$\mathcal{I}(X) = \begin{cases} \frac{\rho^2}{||X-C||^2}(X - C) + C, & X \neq C, \infty \\ \infty & X = C \\ C & X = \infty \end{cases}$$

where $C = (-8, 13)$ and $\rho = 29$ in our case. By my quick calculations, this yields:

$$\mathcal{I}(0,0) = \left(\frac{4864}{233}, \frac{-7904}{233}\right)$$

$\mathcal{I}(12, -8) = (12, -8)$ (this point is on the mirror, so it stays fixed!)

$$\mathcal{I}(\infty) = (-8, 13)$$

$$\mathcal{I}(8, -13) = \left(\frac{1500}{233}, \frac{-4875}{466}\right)$$

7.5: I'm assuming you can make pictures of these in GeoGebra; let me know if you need help with that. As for the answers themselves:

(i) The circle of radius 2 centered at O is sent to the circle of radius $1/2$ centered at O.

(ii) The circle of radius 3 centered at O is sent to the circle of radius $1/3$ centered at O.

(iii) The circle of radius 1 centered at $(0, -1)$ goes through the origin. The origin is sent to ∞, which means the image of our circle will be a line. (Lines contain ∞. Circles don't.) Furthermore, the circle of radius 1 centered at $(0, -1)$ intersects our mirror circle at the points $(\pm\sqrt{3}/2), -1/2)$. [Draw a picture and/or solve the system of equations to see this. Ask me if you're not sure why.] Hence the image is the line through those points, which is the horizontal line $y = -1/2$. You could also find this via formulas in Theorem 7.4.

(iv) The line $x_2 = -1$ includes the point at infinity, so its inversion will include the origin. It also includes the the point $(0, -1)$, which is on the mirror and hence is fixed by the inversion. We could examine some other points, use Theorem 7.4, or draw a diagram like the one I used to prove part (ii) of the theorem, to see this will be the circle with diameter from $(0, 0)$ to $(0, -1)$. That's the circle of radius $1/2$ centered at $(0, -1/2)$.

(v) The line $x_2 = x_1$ (i.e. the line $y = x$) is sent to itself, although the circle inversion is *not* the identity function for points on the line; points on the line but outside the circle are sent to points on the line but inside the circle, and vice versa.

(vi) The circle of radius $5/2$ centered at $(-3, 0)$ does not include the origin, so it will be sent to a circle which does not include the origin. includes the points $(-11/2, 0)$ and $(-1/2, 0)$, which are inverted to $(-2/11, 0)$ and $(-2, 0)$, respectively. These points turn out to be the diameter of the new circle, which is therefore centered at

$$\frac{1}{2}\left((-2/11, 0) + (-2, 0)\right) = (-12/11, 0)$$

and has equation

$$(x + 12/11)^2 + y^2 = (10/11)^2$$

7.13: This amounts to showing that $\mathcal{I}(\mathcal{I}(X)) = X$ for all X. For $X = 0$ and $X = \infty$ this is clear, because

$$\mathcal{I}(\mathcal{I}(0)) = \mathcal{I}(\infty) = 0,$$
$$\mathcal{I}(\mathcal{I}(\infty)) = \mathcal{I}(0) = \infty.$$

For other points, you could make an argument based on the geometric interpretation of circle inversion of X, as a point X' along the ray $\overrightarrow{OX}$ such that $|\overline{OX}| \cdot |\overline{OX'}| = \rho^2$. Or you could shove everything into the formula. It's a mess but it works:

$$\mathcal{I}(\mathcal{I}(X)) = \mathcal{I}\left(\frac{\rho^2}{||X||^2}X\right) = \frac{\rho^2}{\left|\left|\frac{\rho^2}{||X||^2}X\right|\right|^2}\left(\frac{\rho^2}{||X||^2}X\right)$$

$$= \frac{\rho^2}{\frac{\rho^4}{||X||^4}||X||^2}\left(\frac{\rho^2}{||X||^2}X\right) = \frac{\rho^2}{1}\frac{||X||^4}{\rho^4||X||^2}\left(\frac{\rho^2}{||X||^2}X\right)$$

$$= X. \text{ (Whee!)}$$

I'm less concerned about the part of the problem which asks for the consequence of this fact in the proof of Theorem 7.4. Essentially, the fact that $\mathcal{I}(X)$ is an involution helps you reduce the cases; it's what allows us to say "if a line not through O is sent to a circle through O, then a circle through O is sent to a line not through O."

7.14: As suggested in class and email, I'd approach this by moving everything to the left by two units, reflecting across the circle m of radius 1 centered at the origin, and then moving everything back to the right by two units.

(i) The semicircle $(\cos t, \sin t)$, $0 < t < \pi$ is the upper half of the unit circle centered at the origin. Shifted to the left two units, it's the upper half of the circle $(x + 2)^2 + y^2 = 1$. Reflecting across m gives the (upper half of the) circle

$$(x + 2/3)^2 + y^2 = (1/3)^2$$

, i.e. the circle of radius $1/3$ centered at $(-2/3, 0)$. Moving everything to the right two units gives our final answer: the top half of the circle of radus $1/3$) centered at $(4/3, 0)$.

(iv) Moving to the left two units, this becomes the ray $(-1, 0) + t(1, 1)$. This is a line not through the origin, reflected across m to the circle centered at $(-1/2, 1/2)$ with radius $(1/\sqrt{2})$. Because we're reflecting a ray, not an entire line, we don't get the entire circle – it's the portion of the circle for which $y \geq 0$. Moving this back to the right by two units gives our final answer:

$$(x - 3/2)^2 + (y - 1/2)^2 = 1/2, \quad y \geq 0.$$