The goal of this lab assignment is to illustrate through the proof of Theorem 3.39, which is a fundamental result about isometries and allows us to classify them:

> **Theorem 3.39**. Every isometry can be expressed as the composition of three or fewer reflections.

You should carefully read through the background information on this page and the "warm-up" GeoGebra exercises before completing the lab assignment on page 2.

<div align="center">BACKGROUND INFORMATION</div>

We have defined congruence of line segments (equal length) and congruence of angles (equal measure), but we can now use isometries to give an overall definition of congruence. Officially this appears in your text on the first page of Chapter 4:

> **Definition 4.3**. Two subsets of $\mathbb{R}^2$ are *congruent* if there is an isometry which sends one of them to another.

We might be most accustomed to dealing with congruence of triangles, so to rephrase: $\triangle ABC$ is congruent to $\triangle DEF$ if there is an isometry which sends one to the other. In other words, if I can translate, reflect and/or rotate $\triangle ABC$ and get it to exactly match up with $\triangle DEF$, they're congruent. For example, if I reflect every point in $\triangle ABC$ over the line ℓ, the result is a congruent triangle with the same side lengths and angles:

$\triangle ABC$ $\qquad$ $\triangle A'B'C' =$ **reflection of** $\triangle ABC$ **across** ℓ
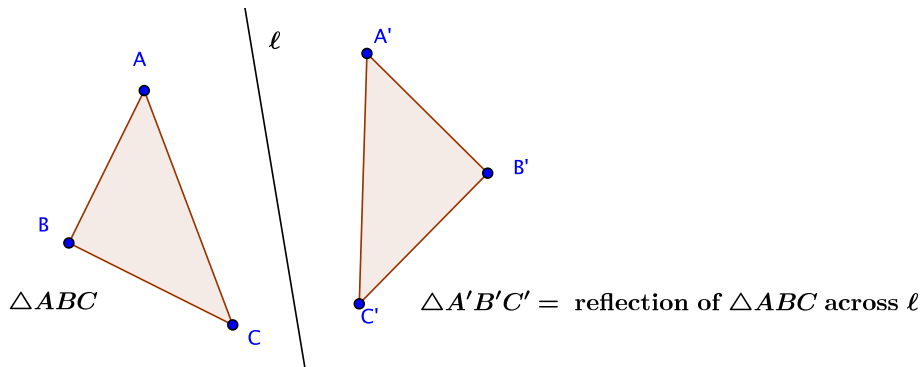
Furthermore, given two congruent triangles, it turns out there is *exactly one* isometry which sends one to the other. The upshot of all this is that we can prove Theorem 3.39 as follows: take any two congruent triangles $\triangle ABC$ and $\triangle DEF$, and show that you can always send $\triangle ABC$ to $\triangle DEF$ by reflecting across one, two or three lines.[1] In the following pictures, for example, $\triangle ABC$ reflects across ℓ to get $\triangle A'B'C$, which is reflected across a second line m to get $\triangle A''B''C''$, which is reflected across a third line n, where it lands on $\triangle DEF$.

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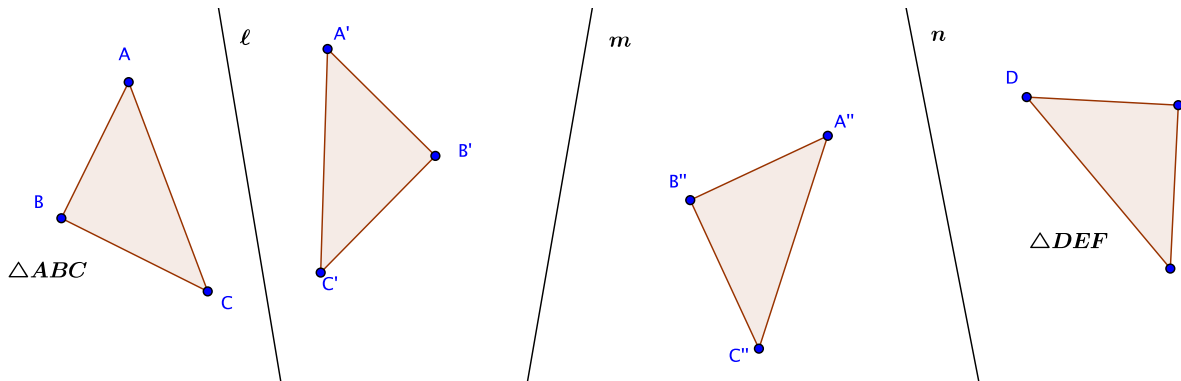
[1]More precisely: given an isometry $\mathcal{U}$, take any triangle $\triangle ABC$ and let $\triangle DEF = \mathcal{U}(\triangle ABC)$. Now we want to show we can send $\triangle ABC$ to $\triangle DEF$ using three or fewer reflections, which would mean we've constructed $\mathcal{U}$ as a composition of those reflections.

If we use the notation R_ℓ for reflection across ℓ, we can describe this as the composition of three reflections:

$$\triangle DEF = R_n \circ R_m \circ R_\ell(\triangle ABC) \quad \text{or} \quad \triangle DEF = R_n(R_m(R_\ell(\triangle ABC)))$$

Finally, recall the following rule, which you noted in class. Make sure you understand/remember why it's true before you continue on to the lab assignment:

> **Rule.** Let ℓ be the perpendicular bisector of $\overline{XY}$. Then the reflection of X across ℓ is Y, and the reflection of Y across ℓ is X.

WARMUP GEOGEBRA EXERCISES

You do not need to turn anything in for this section. Your goal is to learn how to create reflections in GeoGebra.

(1) Open a new GeoGebra window. Create a point A, and then add two or three lines in your window which do not go through A.

(2) Choose the `Reflect` tool (its icon is a blue dot reflected to a red dot across a line) and click on A and a line, in that order. GeoGebra will reflect A across the line and create a point A'. Now reflect A' across a different line to get A''. If you have a third line, you could repeat the process. Note that you might have to zoom out and/or drag the entire picture to see your reflected points.

(3) Drag A around and you should see the reflections move. Same if you drag the lines around. If you're having trouble with this step, check with me during office hours.

(4) Before continuing on to the construction, try the same thing with a triangle instead of a single point A. (If you choose the `Reflect` tool, you can click on the triangle and then the line, and GeoGebra will reflect the entire triangle.)

GEOGEBRA CONSTRUCTION

(1) Open a new GeoGebra window. Hide the axes but turn on the grid.

(2) Create two congruent triangles $\triangle ABC$ and $\triangle DEF$. That means $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, and $\overline{AC} \cong \overline{DF}$. You're free to use any triangles you want, but if you're not feeling creative, here are some suggestions. If you use these you should convince yourself that $\triangle ABC \cong \triangle DEF$.

$$
\begin{array}{ll}
A = (-7, 12) & D = (2, 9) \\
B = (-6, 9) & E = (5, 8) \\
C = (-4, 9) & F = (5, 6)
\end{array}
$$

By default, $\triangle ABC$ (probably `poly1` in GeoGebra's list) will be orange. Make $\triangle DEF$ green so that it's easier to keep track of. You can do this by right clicking on the triangle, choosing Object Properties, and using the Color tab in the resulting window. Alternatively, you can find Object Properties under the Edit Menu, but you'll have to select the correct triangle from the list.

(3) The proof of Theorem 3.39 is *constructive*, meaning it uses step-by-step instructions to construct the three reflections you need to send $\triangle ABC$ to $\triangle DEF$. The first step is to reflect $\triangle ABC$ across the perpendicular bisector of $\overline{AD}$.[2] Create that perpendicular bisector and reflect $\triangle ABC$ across it. That should create $\triangle A'B'C'$, probably called `poly1'` in GeoGebra's list. Make $\triangle A'B'C'$ red so it's easier to spot.

Notice that $D = A'$, i.e. A was reflected onto D. That's because of the **Rule** up above! That's one corner down, and two to go!

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[2]Unfortunately it's hard to get GeoGebra to match the notation in the book, so the names of the points in this lab won't match up with the names in the book's proof of the Theorem.

(4) The next step is to reflect $\triangle A'B'C'$ across the perpendicular bisector of $\overline{B'E}$. Create that bisector. **Notice that the bisector goes through the point** $D = A'$**.** That's because these triangles are all congruent, so $|\overline{DE}| = |\overline{A'B'}|$. In other words, $D = A'$ is the same distance from E and B', so it's on the perpendicular bisector of $\overline{B'E}$. (Think of the warmup problem from class last Wednesday!)

Now reflect $\triangle A'B'C'$ across the bisector of $\overline{B'E}$. That should create $\triangle A''B''C''$, probably called `poly1''` in GeoGebra's list. Color this newest triangle blue so we can keep track of it. Because $D = A'$ was on the bisector, it stayed in place during this reflection. But now B' was reflected onto E because of the **Rule** above. That's two corners down, and one to go!

(5) At this point you can probably see how to reflect your blue triangle $\triangle A''B''C''$ onto the green $\triangle DEF$, but for completeness: create the perpendicular bisector of $\overline{C''F}$. Assuming all has gone well, this bisector will go through D and E. Reflect $\triangle A''B''C''$ across this bisector and verify that the new $\triangle A'''B'''C'''$, called `poly1'''` in GeoGebra's list, is in fact exactly $\triangle DEF$.

(6) Of course, we've just verified Theorem 3.39 in *one* particular case now, so you should click and drag $\triangle ABC$ around to verify that the construction still works. If you're feeling adventuresome, you could completely change A, B, C, D, E, and F to show this works for different congruent triangles $\triangle ABC$ and $\triangle DEF$. Sometimes you don't need all three reflections — $\triangle ABC$ is sent to $\triangle DEF$ by the first one or two!

To receive credit for this assignment, save your file as and email it to me as an attachment by Wednesday, 10/14/15. The email is due by the beginning of class. You will receive full credit if all of these conditions are met:

- The subject of your email is `Math 5335 Lab 2`.
- Your construction is attached to the email with the filename `lastname-5335-lab2.ggb`, where you have replaced the generic `lastname` with your own.
- Your file is a correct and complete construction through step (5) above. In particular, I should be able to drag $\triangle ABC$ around and see that all the reflections update correctly, and eventually land on $\triangle DEF$.