Math 5335
Fall 2018
Exam 1
10/24/18
Time Limit: 75 Minutes

Name (Print): Solutions

This exam contains 7 pages (including this cover page) and 8 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and **put your initials** on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you are applying a theorem, you should indicate this fact**, and explain why the theorem may be applied.
- **Do not trivialize a problem**. If you are asked to prove a theorem, you cannot just cite that theorem.
- **Organize your work** in a reasonable, tidy, and coherent way. Work that is disorganized and jumbled that lacks clear reasoning will receive little or no credit.
- **Unsupported answers will not receive full credit.** An answer must be supported by calculations, explanation, and/or algebraic work to receive full credit. Partial credit may be given to well-argued incorrect answers as well.
- If you need more space, use the back of the pages. **Clearly indicate when you have done this.**

Do not write in the table to the right.

Page	Points	Score
2	11	
3	18	
4	18	
5	25	
6	16	
7	12	
Total:	100	

You may use the following matrices and computations on the exam without defining or proving them.

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \qquad F_\theta = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}, \qquad R_\varphi R_\theta = R_{\varphi+\theta}, \qquad F_\varphi F_\theta = R_{2(\varphi-\theta)}$$

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta), \qquad \sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

Suppose ℓ forms an angle of θ with the horizontal. If $U \parallel \ell$, then $F_\theta U = U$. If $V \perp \ell$, then $F_\theta V = -V$.

1. Let $U = (2, 3)$ and $V = (1, 4)$.

   (a) (1 point) Compute $U \cdot V$.

   $U \cdot V = 2 \cdot 1 + 3 \cdot 4 = 2 + 12 = 14$

   (b) (1 point) Compute $U + V$.

   $U + V = (2+1, 3+4) = (3, 7)$

   (c) (1 point) Compute $\|U\|$.

   $\|u\| = \sqrt{u \cdot u} = \sqrt{4+9} = \sqrt{13}$

2. Let ℓ be the line in the picture below.

   (a) (4 points) Find a parametric equation for ℓ.

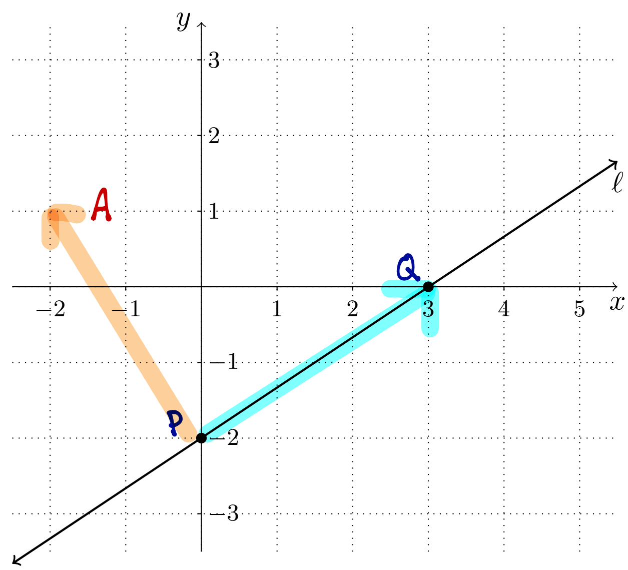
   (Other answers possible)

   $P = (0,-2) \in \ell$

   $Q - P = (3,0) - (0,-2) = (3,2)$

   is a Dir'n Indicator of ℓ

   $P + t(Q-P) = (0,-2) + t(3,2)$



   (b) (4 points) Find a normal equation for ℓ.

   (Other answers possible)

   $(3,2) \parallel \ell$, so $A = (-2,3) \perp \ell$

   $(-2,3) \cdot (X - (0,-2)) = 0.$

3. Recall that $\arccos z = \displaystyle\int_z^1 \frac{1}{\sqrt{1-t^2}}dt$ and we define $\pi = \displaystyle\int_{-1}^1 \frac{1}{\sqrt{1-t^2}}dt$.

(a) (6 points) Let rays p and q emanate from a common vertex, with direction indicators $(-3, 4)$ and $(2, 3)$. Find $|\angle(p, q)|$. An answer with an integral is fine.

We need $\underline{\text{unit}}$ dir'n vectors: $U = \dfrac{(-3,4)}{\|(-1,4)\|} = \dfrac{(-3,4)}{5}$

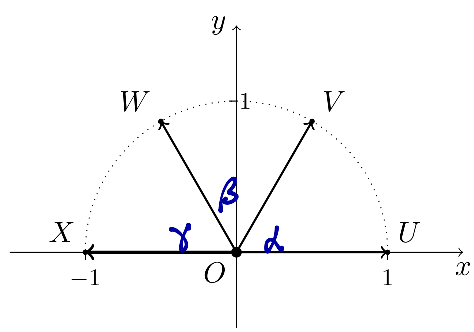
$V = \dfrac{(2,3)}{\|(2,3)\|} = \dfrac{(2,3)}{\sqrt{13}}$ $\qquad U \cdot V = \dfrac{-6+12}{5\sqrt{13}}$

$$|\angle(p,q)| = \int_{U\cdot V}^1 \frac{1}{\sqrt{1-t^2}}dt = \int_{6/5\sqrt{13}}^1 \frac{1}{\sqrt{1+t^2}}dt$$

(b) (6 points) Use calculus to prove that $\arccos(0) = \pi/2$.

$$\arccos(0) = \int_0^1 \underbrace{\frac{1}{\sqrt{1-t^2}}}_{\text{even function}}dt = \frac{1}{2}\int_{-1}^1 \frac{1}{\sqrt{1-t^2}}dt = \frac{1}{2}\pi$$

(c) (6 points) In the picture below, $U = (1, 0)$, $V = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $W = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, and $X = (-1, 0)$. Prove $|\angle UOV| = \pi/3$ using the methods of this course.



$\alpha = \arccos(U\cdot V) = \arccos\left(\frac{1}{2}\right)$

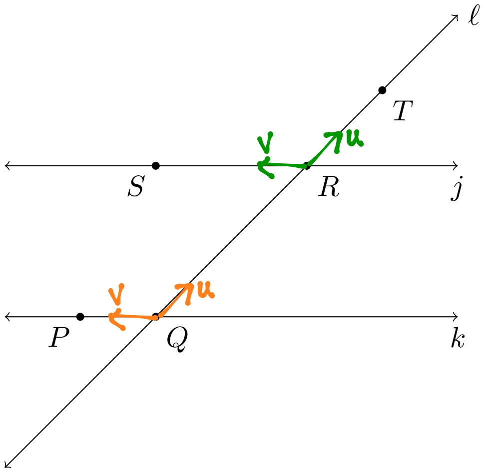
$\beta = \arccos(V\cdot W) = \arccos\left(\frac{1}{2}\right)$

$\gamma = \arccos(W\cdot X) = \arccos\left(\frac{1}{2}\right)$

Thus $\alpha = \beta = \gamma$. Since $\alpha+\beta+\gamma = \pi$, we have

$$\alpha+\beta+\gamma = 3\alpha = \pi \Rightarrow \alpha = \pi/3$$

4. (8 points) In the diagram below, $j \parallel k$. Prove $|\angle PQR| = |\angle SRT|$ using the definitions and methods of this course.



Let $U = \frac{T-R}{||T-R||}$ and $V = \frac{S-R}{||S-R||}$. Then $U \parallel \ell$, hence is a dir'n indicator of $|\angle TRS|$ - and of $|\angle PQR|$.

Similarly, $V \parallel j$ (hence also k) and is the other DI for each angle. Thus

$$|\angle PQR| = \arccos(U \cdot V) = |\angle SRT|$$

5. Prove the following facts about vectors using methods from this class.

(a) (5 points) Given two vectors U and V, prove $||U+V||^2 = ||U||^2 + ||V||^2 + 2\,U \cdot V$.

$$||U+V||^2 = (U+V) \cdot (U+V) = U \cdot U + V \cdot V + U \cdot V + V \cdot U$$
$$= ||U||^2 + ||V||^2 + 2 U \cdot V$$

(b) (5 points) If U and V are unit vectors and $U \cdot V = -1$, then $U = -V$.

$||U|| = ||V|| = 1$. If $U \cdot V = -1$, then by part (a):

$$||U+V||^2 = 1 + 1 - 2 = 0.$$

Thus $||U+V|| = 0 \Rightarrow U+V = 0 \Rightarrow U = -V$.

6. (a) (3 points) Complete the definition: $\mathcal{U}(X)$ is an isometry of $\mathbb{R}^2$ if...

$\|\mathcal{U}(P)-\mathcal{U}(Q)\| \ \forall P,Q \in \mathbb{R}^2$

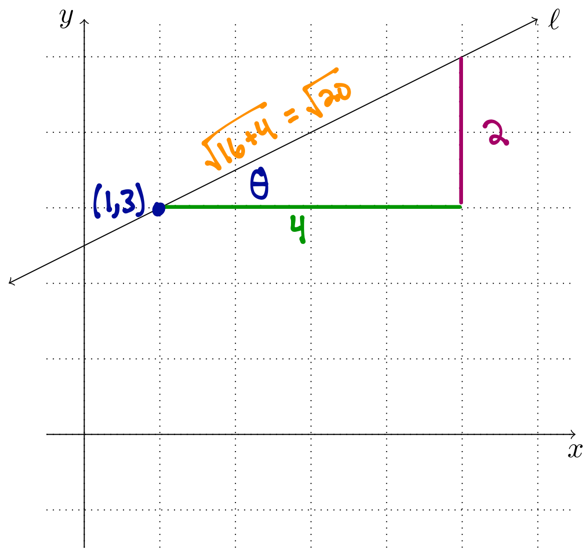
(b) (4 points) Let $\mathcal{T}(X) = X + V$ be translation by the vector V. Prove $\mathcal{T}$ is an isometry.

$$\forall P,Q\in\mathbb{R}^2: \ \|\mathcal{T}(P)-\mathcal{T}(Q)\| = \|P+V-(Q+V)\|$$
$$= \|P+V-Q-V\|$$
$$= \|P-Q\|$$

(c) (6 points) Let $\mathcal{U}$ and $\mathcal{V}$ be isometries of $\mathbb{R}^2$. Prove that the composition $\mathcal{U}\circ\mathcal{V}$ is an isometry.

$$\forall P,Q\in\mathbb{R}^2: \ \|\mathcal{U}(\mathcal{V}(P))-\mathcal{U}(\mathcal{V}(Q))\| = \|\mathcal{V}(P)-\mathcal{V}(Q)\| \quad \text{b/c } \mathcal{U} \text{ is isometry}$$
$$= \|P-Q\| \quad \text{b/c } \mathcal{V} \text{ is isometry}$$

(d) (12 points) Let $\ell = \{(1,3) + t(4,2)\}$, where $t \in \mathbb{R}$. Find the matrix formula $\mathcal{M}_\ell(X)$ for the reflection across the line ℓ. Your answer should include exact values in the matrix, not trig functions, but you do not need to multiply out your entire formula.



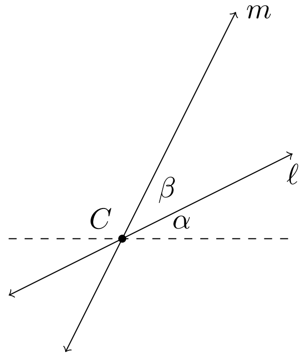
$$\cos\theta = \frac{4}{\sqrt{20}}$$
$$\sin\theta = \frac{2}{\sqrt{20}}$$

$$\cos 2\theta = \left(\frac{4}{\sqrt{20}}\right)^2 - \left(\frac{2}{\sqrt{20}}\right)^2 = \frac{16-4}{20} = \frac{12}{20} = \frac{3}{5}$$
$$\sin 2\theta = 2\left(\frac{4}{\sqrt{20}}\right)\left(\frac{2}{\sqrt{20}}\right) = \frac{16}{20} = \frac{4}{5}$$

$$\mathcal{M}_\ell(X) = F_\theta(X-P)+P = \begin{bmatrix} 3/5 & 4/5 \\ 4/5 & -3/5 \end{bmatrix}\left(X - \begin{bmatrix}1\\3\end{bmatrix}\right) + \begin{bmatrix}1\\3\end{bmatrix}$$

7. In this problem we'll consider the composition of reflections, e.g. $\mathcal{M}_m \circ \mathcal{M}_\ell(X) = \mathcal{M}_m(\mathcal{M}_\ell(X))$. If you use any angles which are not in the digrams, make sure to define and label them.

(a) (8 points) In the diagram below, lines ℓ and m intersect at C, with angles α and β as shown below. The dotted line is horizontal, i.e. parallel to the x-axis. Use the methods of this course to prove $\mathcal{M}_m \circ \mathcal{M}_\ell(X)$ is a rotation about C by an angle of 2β.



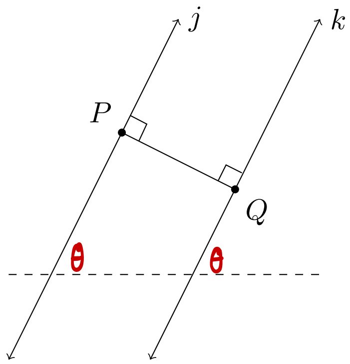
$$\mathcal{M}_m(\mathcal{M}_\ell(X)) = \mathcal{M}_\ell(F_\alpha(X-C)+C)$$

$$= \underbrace{F_{\alpha+\beta}([F_\alpha}(X-C)\underbrace{+C]-C}_{\text{cancel}})+C$$

$$= R_{2\beta}(X-C)+C$$

(By cover page, $F_{\alpha+\beta}\cdot F_\alpha = F_{2(\alpha+\beta-\alpha)} = F_{2\beta}$)

(b) (8 points) In the diagram below, $j \parallel k$, and points $P \in j$ and $Q \in k$ are chosen so that the vector $Q - P$ is perpendicular to both j and l. The dotted line is horizontal, i.e. parallel to the x-axis. Use the methods of this course to prove $\mathcal{M}_k \circ \mathcal{M}_j$ is a translation by $V = 2(Q-P)$.



$$\mathcal{M}_k(\mathcal{M}_j(X)) = \mathcal{M}_\ell(F_\theta(X-P)+P)$$

$$= F_\theta([F_\theta(X-P)+P]-Q)+Q$$

$$= \underbrace{F_\theta F_\theta}_{=I}(X-P) + \underbrace{F_\theta(P-Q)}_{=-(P-Q)} + Q$$

(cover page)

$$= X-P-P+Q+Q$$

$$= X+2Q-2P$$

$$= X+2(Q-P)$$

8. (12 points) Indicate whether each statement is **True** or **False** by circling the appropriate answer. Justify your answer with definitions, theorems and methods from this course. (If false, **be specific** with your explanation; e.g. tell me what part of a definition or theorem is not satisfied, or give an example to show the statement is false, etc.)

(a) For any A, B and C, the barycentric coordinates of the origin $(0,0)$ are always $(0,0,0)^{\triangle ABC}$.

True **False**

$(0,0,0)^{\triangle ABC}$ not valid BC's; $0+0+0 \neq 1$

(b) If $\triangle ABC \cong \triangle A'B'C'$, there exist two different isometries which send A to A', B to B', and C to C'.

True **False**

$\exists$ unique such isometry, by Chapter 4.

(Thms 4.17, 4.18)

(c) Any isometry can be constructed as the composition of two or fewer reflections.

True **False**

Three might be required. (Lab 3, for example - or any glide reflection)