Math 5335
Fall 2017
Exam 2
12/5/18
Time Limit: 80 Minutes

Name (Print): KEY

This exam contains 8 pages (including this cover page) and 13 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and **put your initials** on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you are applying a theorem, you should indicate this fact**, and explain why the theorem may be applied.
- **Do not trivialize a problem**. If you are asked to prove a theorem, you cannot just cite that theorem.
- **Organize your work** in a reasonable, tidy, and coherent way. Work that is disorganized and jumbled that lacks clear reasoning will receive little or no credit.
- **Unsupported answers will not receive full credit.** An answer must be supported by calculations, explanation, and/or algebraic work to receive full credit. Partial credit may be given to well-argued incorrect answers as well.
- If you need more space, use the back of the pages. **Clearly indicate when you have done this.**

Do not write in the table to the right.

Page	Points	Score
2	14	
3	15	
4	12	
5	15	
6	21	
7	10	
8	13	
Total:	100	

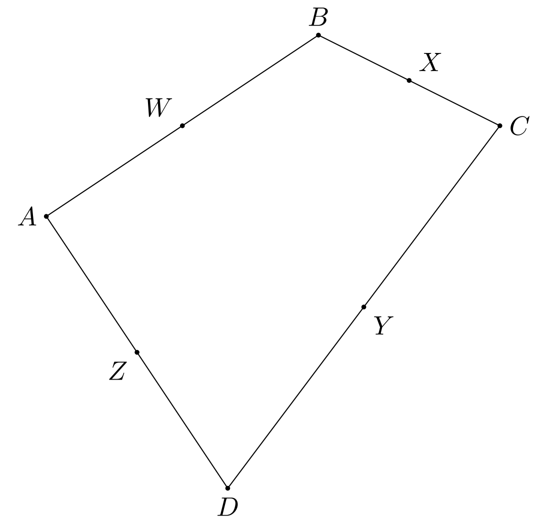
You may use the following results on the exam without defining or proving them.

The distance between points (a, b) and (a, d) in the Poincaré Half Plane is $|\ln(d/b)|$.

The distance between points P_1 and P_2 on the line $(x-\omega)^2 + y^2 = \rho^2$, with angles $t_1 = |\angle(\omega+\rho, 0)(\omega, 0)P_1|$ and $t_2 = |\angle(\omega+\rho, 0)(\omega, 0)P_2|$ is

$$\ln\left[(\csc t_2 - \cot t_2) \,/\, (\csc t_1 - \cot t_1)\right]$$

1. (8 points) Let $ABCD$ be a convex quadrilateral which is not necessarily a parallelogram, trapezoid, or other familiar shape. Let W, X, Y and Z be the midpoints of the sides, as shown in the generic diagram below. Prove that $WXYZ$ is a parallelogram.



$W = \frac{A+B}{2}$ $X = \frac{B+C}{2}$ $Y = \frac{C+D}{2}$ $Z = \frac{A+D}{2}$

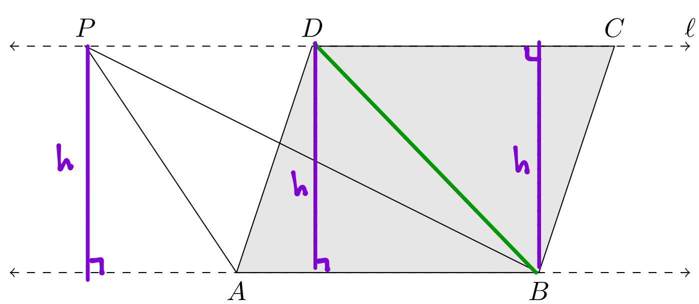
Many approaches possible, including:

$W+Y = \frac{1}{2}(A+B+C+D)$

$X+Z = \frac{1}{2}(A+B+C+D)$

Since $W+Y = X+Z$, $WXYZ$ is a $\|$'gram

2. (6 points) Suppose a parallelogram $ABCD$ and triangle $\triangle ABP$ are constructed on the same base $\overline{AB}$, and C, D and P are all on the same line ℓ, which is parallel to $\overleftrightarrow{AB}$. Prove the area of $ABCD$ is twice the area of $\triangle ABP$.



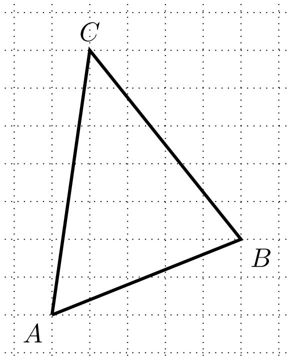
(many approaches possible)

The $\|$'gram can be split into 2 $\triangle$'s with same base ($\overline{AB} \cong \overline{CD}$ b/c $ABCD$ is a $\|$'gram). All 3 $\triangle$'s have same height h.

Thus $\underbrace{\|\triangle ABD\| = \|\triangle BCD\|}_{\|ABCD\|} = \|\triangle ABP\|$

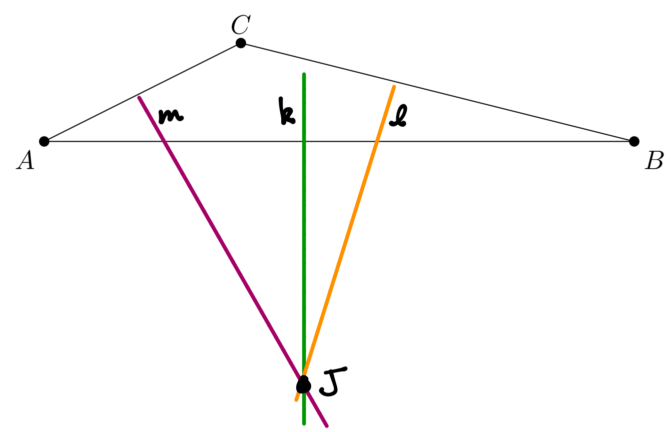
Hence $\|ABCD\| = 2\|\triangle ABC\|$.

3. (3 points) Let $A = (1,1)$, $B = (6,3)$ and $C = (2,8)$. Find the centroid of $\triangle ABC$, in rectangular coordinates.



$$G = \left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right)^{\Delta} = \tfrac{1}{3}(A+B+C) = \tfrac{1}{3}(9,12) = (3,4)$$

4. (6 points) Prove that the perpendicular bisectors of (the sides of) $\triangle ABC$ all intersect in a point J which is equidistant to all three vertices. (You can use the diagram below for convenience but your argument must apply to all triangles.)



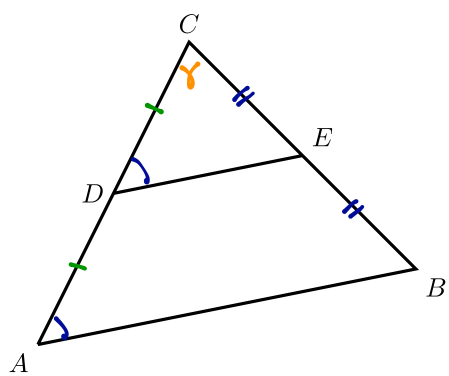
⚠ No two sides of $\triangle ABC$ are $\parallel$ — hence their bisectors are not $\parallel$, and must intersect

Let k = perp. bisector of $\overline{AB}$
ℓ = —— " —— $\overline{BC}$
m = —— " —— $\overline{AC}$

Let $J = k \cap \ell$. By prop of $\perp$ bisectors,
$J \in k \Rightarrow |\overline{AJ}| = |\overline{JB}|$
$J \in \ell \Rightarrow |\overline{CJ}| = |\overline{JB}|$
Hence $|\overline{JA}| = |\overline{JB}| = |\overline{JC}|$ so J equidistant to A, B, C.

In particular, $|\overline{JB}| = |\overline{JC}| \Rightarrow J \in m$, so all 3 $\perp$ bisectors are concurrent at J.

5. (6 points) Given $\triangle ABC$, let D and E be the midpoints of $\overline{AC}$ and $\overline{BC}$ as shown. Using any appropriate methods from the course, prove $\overline{DE} \parallel \overline{AB}$ and $|\overline{DE}| = \frac{1}{2}|\overline{AB}|$



Multiple approaches possible.

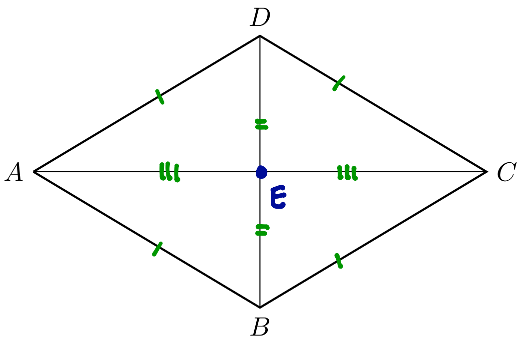
$\triangle ACB \simeq \triangle DCE$ by SAS Similarity.
$\frac{|\overline{CD}|}{|\overline{CA}|} = \frac{1}{2} \Rightarrow \frac{|\overline{DE}|}{|\overline{AB}|} = \frac{1}{2}$. Also, $\angle CAB \cong \angle CDE$.

By corresponding angles, $\overline{DE} \parallel \overline{AB}$.

OR $D = \frac{A+C}{2}$, $E = \frac{B+C}{2} \Rightarrow E - D = \frac{B-A}{2} \Rightarrow E - D \parallel B - A$

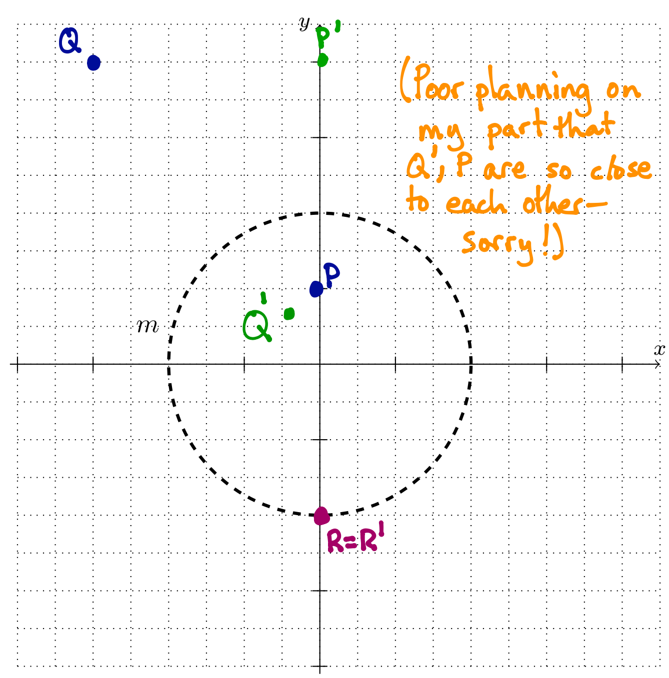
and $\|E - D\| = \frac{1}{2}\|B - A\|$

6. (6 points) Let ABCD be a rhombus. Prove the diagonals are perpendicular.



A rhombus is a ||'gram, so the diagonals bisect each other. All four Δ's are congruent by SSS. Hence there are 4 congruent angles at E which combine to measure 2π. Thus each of those angles is $\frac{\pi}{2}$ and the segments are $\perp$.

7. (6 points) Let $m = \{\|X\| = 2\} = \{(x, y) : x^2 + y^2 = 4\}$. Find the coordinates of the reflection of each of the following points across m. Graph the original points P, Q and R, and their reflections P', Q', and R'. *Note that the gridlines are drawn every half unit in this picture.*



- $P = (0, 1)$

$$\iota(P) = \frac{4}{\|(0,1)\|^2}(1,0) = 4(0,1) = (0,4)$$

- $Q = (-3, 4)$

$$\iota(Q) = \frac{4}{\|(-3,4)\|^2}(-3,4) = \frac{4}{25}(-3,4)$$

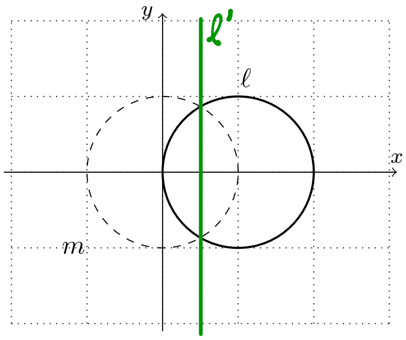
$$= \left(-\frac{12}{25}, \frac{16}{25}\right)$$

- $R = (0, -2)$

$\iota(0,-2) = (0,-2)$ (it's on the mirror)

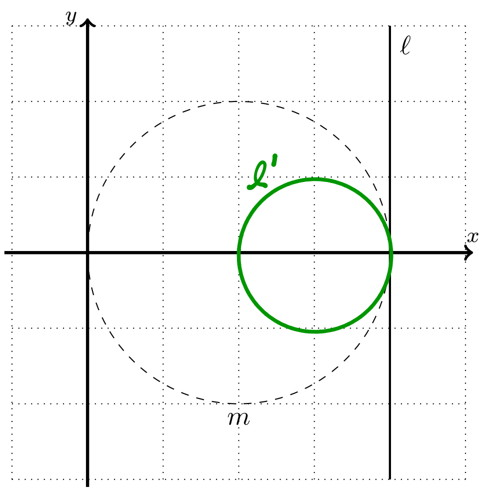
8. In each part below, sketch ℓ', the reflection (i.e. inversion) of ℓ across the mirror m. Also give the equation for ℓ'.

(a) (5 points) $m : x^2 + y^2 = 1$ and $\ell : (x-1)^2 + y^2 = 1$.



m and ℓ intersect at $\left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$, so those pts are in ℓ'. ℓ contains center of $m \Rightarrow \infty \in \ell' \Rightarrow \ell'$ a line. Thus ℓ' is $x = \frac{1}{2}$

(b) (5 points) $m : (x-2)^2 + y^2 = 4$ and $\ell : x = 4$.



$m \cap \ell = (4,0)$ is fixed, so $(4,0) \in \ell'$

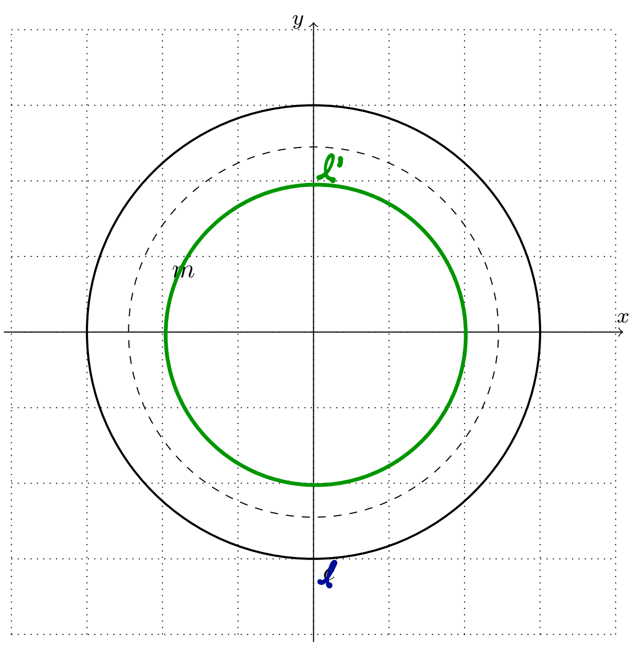
$\infty \in \ell \Rightarrow$ center $(2,0) \in \ell'$

$(2,0) \notin \ell$, so $\infty \notin \ell' \Rightarrow \ell'$ a circle, not a line.

(Also, $(4,2) \in \ell$ reflects to $(3,1)$)

$(x-3)^2 + y^2 = 1$

(c) (5 points) $m : x^2 + y^2 = 6$ and $\ell : x^2 + y^2 = 9$.



All points $X \in \ell$ have $\|X\| = 3$.

They're sent to points X' s.t.

$\|X\| \cdot \|X'\| = \rho^2$

$3\,\|X\| = 6$

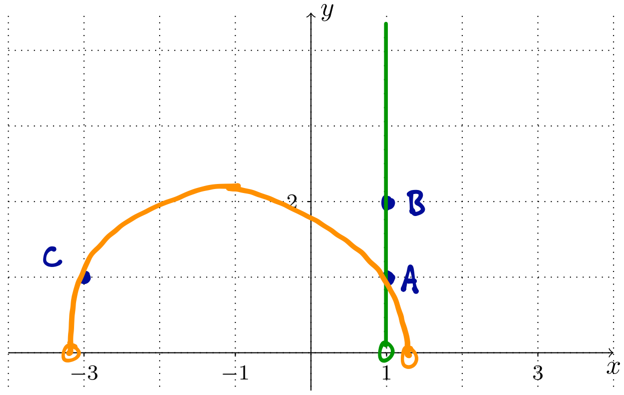
$\|X\| = 2$

Hence ℓ' is $x^2 + y^2 = 4$

On this page, all points, lines, segments and distances are in the **Poincaré Half Plane**.

9. Let $A = (1, 1)$, $B = (1, 2)$ and $C = (-3, 1)$.

(a) (10 points) Sketch the (Poincaré) lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$. Find the equation for each line.



$\overleftrightarrow{AB}$: $x=1$

Center of $\overleftrightarrow{AC}$ must be $\left(\frac{1-3}{2},0\right)=(-1,0)$

radius is $\|A-(-1,0)\|=\sqrt{5}$

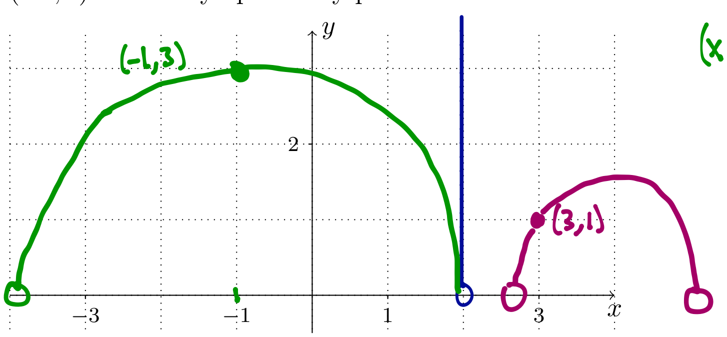
$\overleftrightarrow{AC}$: $(x+1)^2+y^2=5$

(b) (3 points) Find the length of segment $\overline{AB}$.

W/ formula on front: $d(A,B)=\ln\left(\frac{2}{1}\right)=\ln 2$ (or $\left|\ln\frac{1}{2}\right|$)

10. Let ℓ be the line directed by $\mathfrak{A} = (2, 0)$ and $\mathfrak{B} = (\infty, 0)$.

(a) (5 points) Sketch ℓ on the grid below. Then sketch and give an equation for a line m which contains the point $(-1, 3)$ and is asymptotically parallel to ℓ.



$(x+1)^2+y^2=9$

(b) (3 points) Give an equation for a line k containing $(3, 1)$ which is ultra parallel to ℓ, or explain why none exists.

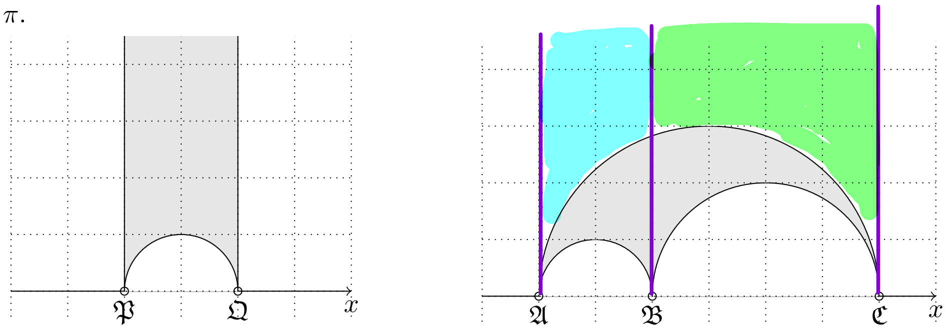
(Many answers possible)

$(x-4)^2+y^2=2$

(3,1) satisfies the eqn, and PDI's are $(4\pm\sqrt{2},0)$. Note that $4-\sqrt{2}\approx 4-1.414>4-1.5=3.5>2$

On this page, all points, lines, segments, triangles and areas are in the **Poincaré Half Plane**.

11. (a) (5 points) The area of any triply asymptotic triangle with Poincaré Direction Indicators $\mathfrak{P} = (p, 0)$, $\mathfrak{Q} = (q, 0)$ and $\mathfrak{R} = (\infty, 0)$, as shown on the left, is π. Use this fact to prove the area of the triply asymptotic triangle $\triangle\mathfrak{A}\mathfrak{B}\mathfrak{C}$ on the right is also π.



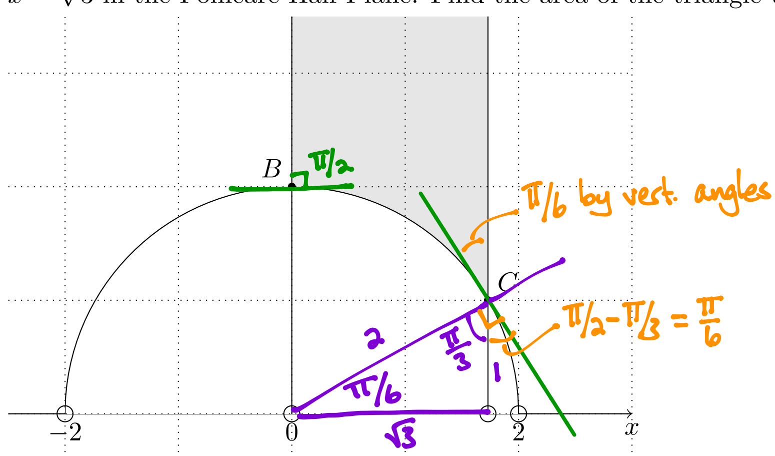
Let $\mathfrak{R} = (\infty, 0)$

$$\|\triangle\mathfrak{A}\mathfrak{B}\mathfrak{C}\| = \|\triangle\mathfrak{A}\mathfrak{B}\mathfrak{R}\| + \|\triangle\mathfrak{B}\mathfrak{C}\mathfrak{R}\| - \|\triangle\mathfrak{A}\mathfrak{C}\mathfrak{R}\|$$

$$= \pi + \pi - \pi$$

$$= \pi$$

(b) (5 points) The picture below shows a singly asymptotic triangle $\triangle\mathfrak{A}BC$ formed by the lines $x^2 + y^2 = 4$, $x = 0$ and $x = \sqrt{3}$ in the Poincaré Half Plane. Find the area of the triangle using any valid method.



$$\|\triangle\mathfrak{A}BC\| = \pi - \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$$

12. (5 points) Recall the incidence axioms from Chapter 10:

I.1 For any two distinct points P and Q, there exists a unique line that is incident with both P and Q.

I.2 Every line is incident with at least two points.

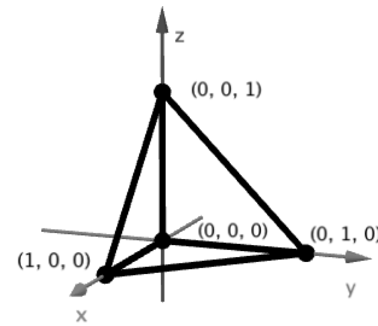
I.3 There exist three points such that no line is incident with all three.

Define a *4 point geometry* as follows. The *points* are the ordered triples $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. As it happens those are the four vertices of a unit tetrahedron, as shown in the picture below. A line is defined to be a set of three points which from the vertices of a triangle in the tetrahedron; for example, $\{(0,0,0),(1,0,0),(0,1,0)\}$ is a line because those points are the vertices of the triangle on the bottom of the tetrahedron.

Does this 4 point geometry satisfy the incidence axioms? Justify your answer.

No it doesn't satisfy I.3.

Any 3 pts form a line



13. For each of the following statements, indicate whether it is **True** or **False** by circling the corresponding answer. Briefly justify your answer.

(a) (4 points) Lines $x = \lambda$ and $x = \mu$ in the Poincaré Half Plane can never be ultra parallel.

True **False**

They always share PDI $(\infty, 0)$

(b) (4 points) Conformal affinities preserve the area of quadrilaterals.

True **False**

Conformal affinities scale, which affects area.