This is an open-book, -library, -internet take home exam. However, standard academic honesty rules apply. If you make use of an outside resource, keep the following in mind if you want to receive credit for your solution:

- Cite the resource as part of your solution. *Don't claim somebody else's work as your own.*
- You must still write the solution in your words. I'm interested in whether you understand the ideas, not whether you can transcribe somebody else's work. Here's a good guideline: wait a few hours; if you can write out your answer without using the outside resource, then you understand the ideas.
- Your solution must be consistent with the definitions and methods of this course. For example, Wikipedia has a page about the Poincaré Half Plane, but the descriptions, notations and methods on that page are totally different than our book's. If you hand in a solution making use of Riemannian metrics, I can't count it for credit unless you also develop the subject of Reimannian metrics. So you're better off using our text book and notes from class.

Please take these guidelines seriously. I can use Google too.

You may not collaborate with others on the exam; I am the only person you are allowed to consult. That includes people online; "open-internet" means you can make use of resources which are already out there – **not** posting on Stack Exchange (Quora, Yahoo Answers, etc.) for help. You can ask me questions during office hours, or you can email me at any time during the day.

Our final exam is scheduled for 8:00 to 10:00 a.m., Tuesday, December 17. I'll give you an extra day and make the take home final due on Wednesday, December 18. (I'd gladly give you more time, but I have my own deadline for when I need to grade finals and submit grades, and the clock starts ticking at our scheduled exam time on 12/17. If the deadline were any later, I wouldn't have time to finish.) I'll hold office hours on Friday, 12/13 from 1:00-3:00pm and Monday, 12/17 from Noon-1:00pm. I'll also be available at other times if those office hours don't work for you; send me email to arrange a time to meet.

Due: Wednesday, 12/18/2019 at noon in my mailbox in Vincent 107.

You should plan to spend at least as much time on this exam as you would have spent studying for (and taking) an in-class final. As always, you should explain your work, writing complete sentences with reasonably correct grammar. A good rule of thumb is that the work you hand in for this exam **should not be your first draft**. Figure out the problem on another sheet of paper, organize your thoughts, and then write out your solution.

You should answer questions in the spirit they are intended, using the appropriate methods. For example, when you're asked to find the measure of an angle using the definition or methods in Chapter 3, you should not argue that the angle happens to be the interior angle of a regular hexagon, and then cite Corollary 8.13 to say its measure is $4\pi/6 = 2\pi/3$. You can always ask me if you're not sure whether a method is appropriate or not! (It helps to start the exam early enough so you have time to ask questions.)

The exam covers Euclidean/Vector Geometry and Poincaré Half Plane Geometry, with more of an emphasis on the latter since you've already been tested more on Euclidean/Vector Geometry.

Problems

Euclidean Geometry. In the following problems you should interpret all terms in the sense of the Euclidean vector geometry we developed up through Chapter 9. (40 Points Total)

Problem E.1: (15 Points) Use the methods of Chapter 3 to find the measure of the angle $\angle(0,3)(0,0)(-\sqrt{6},-\sqrt{2})$. You may use the definition that $\arccos(-1) = \int_{-1}^{1} \frac{1}{\sqrt{1-t^2}}\,dt$ equals π, but you must develop any other angle measures that you use.

Problem E.2: (15 Points) Let $ABCD$ and $WXYZ$ be congruent quadrilaterals in the Euclidean plane. Describe a sequence of 3 or fewer reflections which will be guaranteed to map $ABCD$ to $WXYZ$. Justify why your sequence works; illustrations will be helpful for full credit!

Problem E.3: (10 Points) A quadrilateral $ABCD$ is called *cyclic* if all four of its vertices are on a single circle in the Euclidean plane. Given three noncollinear points A, B and C, describe the set of all points D such that $ABCD$ is cyclic and simple. Justify your answer.

Poincaré Half Plane Geometry. The rest of this exam covers geometry of the Poincaré Half Plane. For the remaining problems, you should interpret all terms (lines, angles, triangles, distances, isometries, etc.) in the context of Poincaré Half Plane Geometry. (60 Points)

Problem P.1: (5 Points) Find four Poincaré lines in the half plane, each of which is asymptotically parallel to both $(x-3)^2+y^2=4$ and $(x+1)^2+y^2=1$. Give the equations of your lines and show them in a sketch.

Problem P.2: (15 Points) Draw a careful, accurate picture of the triangle with vertices $(4,2)$, $(8,2)$ and $(8,\sqrt{20})$. Then find its area without directly computing an integral. Use exact values in your work, but at the end also compute your area to at least five decimal places. *(Hint: two angles in the triangle are very nice. The third isn't. Alas, we can't always have everything we want!)*

Problem P.3: (15 Points) Prove the area of a hyperbolic triangle $\triangle ABC$ is $\pi-\alpha-\beta-\gamma$. You may use Lemmas 11.14, 11.16 and 11.17.

Problem P.4: (20 Points) In class we proved that $\mathcal{M}_\lambda$ and $\mathcal{M}_{\omega,\rho}$ are isometries of the Poincaré Half Plane. Furthermore, as in Euclidean geometry, the composition of Poincaré isometries is a Poincaré isometry. Use these facts to prove the following, writing $d(P,Q)$ for the Poincaré distance between two points P and Q.

(a) (5 Points) Let $\mathcal{T}_s(x,y)=(x+s,y)$ be the function which translates points in the Poincaré Half Plane horizontally by s. Prove that $\mathcal{T}_s$ is an isometry of the Poincaré Half Plane, i.e. prove that it preserves the Poincaré distance between points.

(b) (10 Points) Let $\mathcal{D}_{0,s}(x,y)=(sx,sy)$ be the function which dilates the Poincaré Half Plane by scaling radially away from the origin by a factor of s Prove that $\mathcal{D}_{0,s}$ is an isometry of the Poincaré Half Plane.

(c) (5 Points) Prove that $\mathcal{D}_{\omega,s}$ is an isometry of the Poincaré Half Plane, where $\mathcal{D}_{\omega,s}$ performs the same scaling, but from $(\omega,0)$ instead of $(0,0)$.

Problem P.5: (5 Points) *This is intentionally a more difficult problem, but is only worth 5 points; don't spend time on this until/unless you're certain that the rest of your exam is in good shape.)*

Prove the triangle inequality holds in the Poincaré Half Plane: given points P, Q and R,

$$d(P,Q)+d(Q,R)\geq d(P,R)$$

where $d(A,B)$ is the Poincaré distance from A to B.

Hint: There are a lot of ways P, Q and R can be arranged with respect to each other. I wouldn't recommend approaching this problem by proving each of those individual cases. Instead, transform the general case into a more tractable situation with fewer things to check. I can save you the trouble of searching online: the proofs you'll find involve metrics, hyperbolic circles, and other concepts we haven't covered. Instead, the ideas behind the "Worthwhile Reflection" sheet from class, and Proposition 12.9 in the book, would suffice to write out a proof.