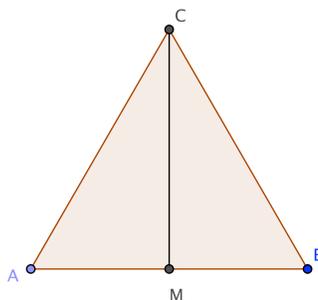


The following is a non-comprehensive list of solutions to the exam problems. In some cases I may give an answer with just a few words of explanation. On other problems the stated solution may be complete. As always, feel free to ask if you are unsure of the appropriate level of details to include in your own work.

As you noticed, the exam was fairly triangle-heavy. There are two reasons for that. First, when the homework from chapters 7 and 7 was delayed, I made an intentional decision to remove some of the test questions covering material from those chapters; I had to fill the resulting void with questions from chapters 4 and 5. Second, triangles are fundamental to everything we've done, even with polygons – we can split polygons into triangles, etc. This trend is noticeable in problem 6(b) on the exam, which is straight out of the chapter 6 exercises, and yet is most easily solved with techniques from chapter 4.

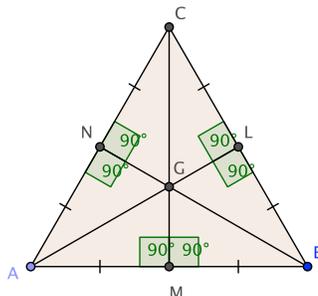
These solutions were typed up while basking in the glory of the Minnesota Wild building a 4-0 lead on the Washington Capitals, followed by the despair of watching them cough up three goals late in the game. Needless to say there were a few typos. I think I've caught them all, but please let me know if you spot any errors and I'll update things as soon as possible.

- (1) (a) The orthocenter is at the origin; you proved this has to be the case in your homework (Chapter 4 #8). The circumcenter is  $(2, 3/2)$ . [Note the similarity to Problem 6(a) on this exam!].
  - (b) The Euler line goes through the orthocenter and circumcenter, so its equation is  $y = \frac{3}{4}x$ . The angle bisector at vertex  $A$  is on the line  $y = x$ , which only intersects the Euler line at  $A$ . Because the incenter is on  $y = x$  but in the interior of the triangle, it cannot be on the Euler line.
  - (c)  $\frac{1}{2}bh = \frac{1}{2} \cdot 4 \cdot \frac{3}{2} = 3$ .
- (2) This problem was extremely similar to multiple homework problems, not to mention the sample problem I did at the beginning of class two days before the exam.
    - (a) Draw the median from  $C$  and denote its intersection with the opposite side by  $M$ . Thus  $\overline{AM} \cong \overline{MB}$ . Hence by SSS (check this!) we have  $\triangle ACM \cong \triangle BCM$ : In particular  $|\angle ACM| = |\angle BCM|$  so that the median  $\overline{CM}$  is also a bisector. Furthermore,  $|\angle AMC| = |\angle BMC|$  and the two angles must add to  $\pi$ , so each of them must be  $\pi/2$ . This immediately implies that  $\overline{CM}$  is also an altitude and is part of the perpendicular bisector of  $\overline{AB}$ .



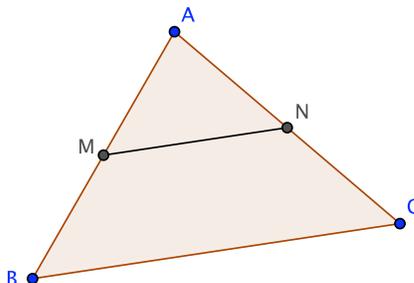
We could have done the same work using the bisectors from the other two vertex angles, so we conclude that the medians, altitudes, bisectors and perpendicular bisectors through the angles are all the same. Hence all of these lines (or line segments) intersect at the same point, and the centroid, orthocenter, incenter and circumcenter all coincide.

- (b) In the diagram below I've filled in some of the information we know. The medians of an equilateral triangle separate the three sides into six congruent segments. We also know the medians are altitudes, and I've labeled the right angles at the feet.



We can also say that  $\overline{AG} \cong \overline{BG} \cong \overline{CG}$ ; this follows either from the definition of perpendicular bisector or from the definition of angle bisector (and the medians are both). Hence by SSA for right triangles, the six triangles inside  $\triangle ABC$  are congruent.

- (3) I intended people to use (a) to show that  $\triangle ABC$  and  $\triangle AMN$  are similar and use that to solve (b). Unfortunately, most people immediately claimed that the two triangles were similar by stating (for example) that  $|\angle AMN| = |\angle ABC|$  and used similarity to prove the segments were parallel. The problem is that you can't just claim  $|\angle AMN| = |\angle ABC|$  – that's a consequence of the line segments being parallel!



- (a)  $M = (A+B)/2$  and  $N = (A+C)/2$ , so  $N - M = \frac{A+C-A-B}{2} = \frac{1}{2}(C-B)$ . Hence  $N - M$  is parallel to  $C - B$ , which is equivalent to  $\overline{MN} \parallel \overline{BC}$ .
- (b)  $\overline{MN} \parallel \overline{BC}$  implies that  $|\angle AMN| = |\angle ABC|$  and  $|\angle ANM| = |\angle ACB|$ , so the two triangles  $\triangle AMN$  and  $\triangle ABC$  are similar. Using that and the Law of Sines,

$$\frac{|\overline{AB}|}{|\overline{BC}|} = \frac{\sin |\angle ACB|}{\sin |\angle BAC|} = \frac{\sin |\angle ANM|}{\sin |\angle MAN|} = \frac{|\overline{AM}|}{|\overline{MN}|}$$

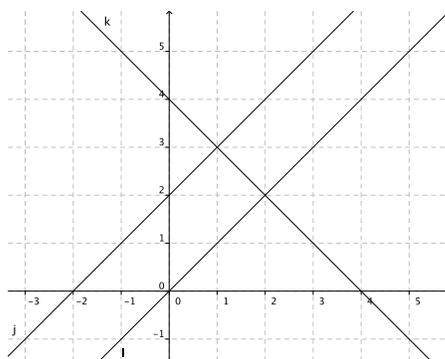
A little cross multiplication finishes the problem.

- (4) This was nearly an exact copy of exercise 30 from chapter 7, but I made the angles much easier so you didn't need to remember any double angle formulas for cosine and sine to construct the matrix formulas. I'd suggest following the homework solutions for that problem (and #6,7 from chapter 7) while looking over these solutions.

- (a) The mirror has a slope of 1, which means it forms an angle of  $\theta = \pi/2 = 45^\circ$  with the horizontal. The point  $P = (0, 2)$  is on the mirror.

$$\begin{aligned} \mathcal{M}_1(X) &= F_\theta(X - P) + P \\ &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \left( \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right) + \begin{bmatrix} 0 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} -2 \\ 2 \end{bmatrix} \end{aligned}$$

- (b) Here we use our previous knowledge about the composition of reflections.



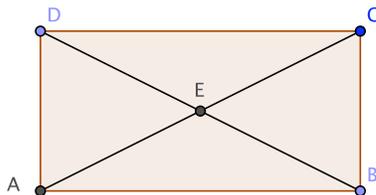
$j$  and  $k$  intersect at a right angle at  $(1, 3)$ , so  $\mathcal{M}_k \circ \mathcal{M}_j(X)$  will be a rotation by  $2 \cdot \pi/2 = \pi$  about the point  $(3, 3)$ , which has the formula

$$\mathcal{M}_k \circ \mathcal{M}_j(X) = X + 2(1, 3)$$

$j$  and  $l$  are parallel, and a vector  $U$  which goes straight from  $j$  to  $l$  would be given by (for example)  $U = (1, 1) - (0, 2) = (1, -1)$ . Hence we get a translation:

$$\mathcal{M}_l \circ \mathcal{M}_j(X) = X + 2U$$

- (5) On your homework you did the companion problem to this one, and I listed it as being an important one to study. (This exact problem was also listed as a review problem). Strangely enough this version seems easier than the homework problem where you start with a rhombus, especially when trying to show (iii) implies either of the other two conditions.

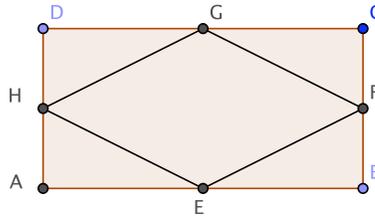


Keep in mind that we know  $ABCD$  is a parallelogram so we can make free use of the following facts:

- The diagonals bisect each other:  $\overline{DE} \cong \overline{EB}$ ,  $\overline{AE} \cong \overline{EC}$ .
- Opposite sides are congruent:  $\overline{AB} \cong \overline{DE}$ ,  $\overline{BC} \cong \overline{AD}$ .

Now we can prove the desired facts.

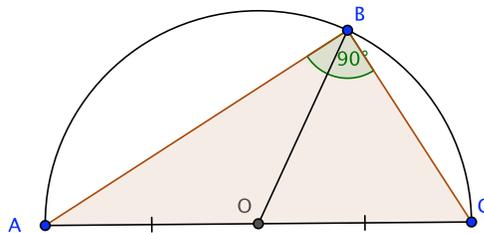
- Because the angles at  $A$ ,  $B$ ,  $C$  and  $D$  are equal and (as with any convex quadrangle) add to  $2\pi$ , they must be right angles. Hence by SAS,  $\triangle DAB \cong \triangle CBA$ , so  $\overline{BD} \cong \overline{AC}$ .
- Now by SSS,  $\triangle DAB \cong \triangle CBA \cong \triangle ADC \cong \triangle BCD$ . Thus all four vertex angles must have equal measure.
- Let's update our picture to include the midpoints.



Because opposite sides are congruent (see above) and the angles must all be right angles [as in part (a)], SAS tells us that all four triangles formed by our diagram are congruent. In particular the four sides of  $EFGH$  are congruent, so it's a rhombus.

- Let's again focus on the four triangles in the diagram. The sides which are part of  $EFGH$  are all congruent since we're assuming (iii). The other sides (e.g.  $\overline{AH}$  and  $\overline{BF}$ , or  $\overline{AE}$  and  $\overline{DG}$ ) are congruent because we have midpoints of a parallelogram whose opposite sites are congruent. So SSS tells us the four triangles are congruent, proving [as in part (b)] that the four vertex angles are equal.
- (6) (a) Following the hint, let  $A = (a, 0)$ ,  $B = (0, b)$  and  $C = (0, 0)$ . The perpendicular bisectors of  $\overline{AC}$  and  $\overline{BC}$  are the lines  $x = a/2$  and  $y = b/2$ , respectively. These intersect at the point  $(a/2, b/2)$  which just happens to be the midpoint of  $\overline{AB}$ , so this point is on the perpendicular bisector of that edge. Hence this point is the intersection of all three perpendicular bisectors and is the circumcenter.

- This was a review problem and is very similar to the Theorem of Thales we discussed in class. It turns out the circle is somewhat of a red herring. Here's what it amounts to: if  $O$  is the midpoint of  $\overline{AC}$ , then we want to prove  $\overline{OB}$  is congruent to the radii  $\overline{OA}$  and  $\overline{OC}$ .



You *could* label angles and try to use triangle congruence theorems. The faster way is to use part (a) to say  $O$  is the circumcenter of  $\triangle ABC$  and is therefore the same distance from each vertex of the triangle.